Lecture 05: Screw Theory

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Modern Robotics Chapters 3.3-3.4

Quick Recap of Past Lectures

1 We can define rotation matrices with a matrix exponential: $R = e^{[\hat{\omega}]\theta}$
Where $e^{[\hat{\omega}]\theta}$ is computed with Rogrigues's Formula
$\hat{\omega}\theta$ are referred to as exponential coordinates

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For transformations (rotations + translations), we need angular and linear (spatial) velocities!
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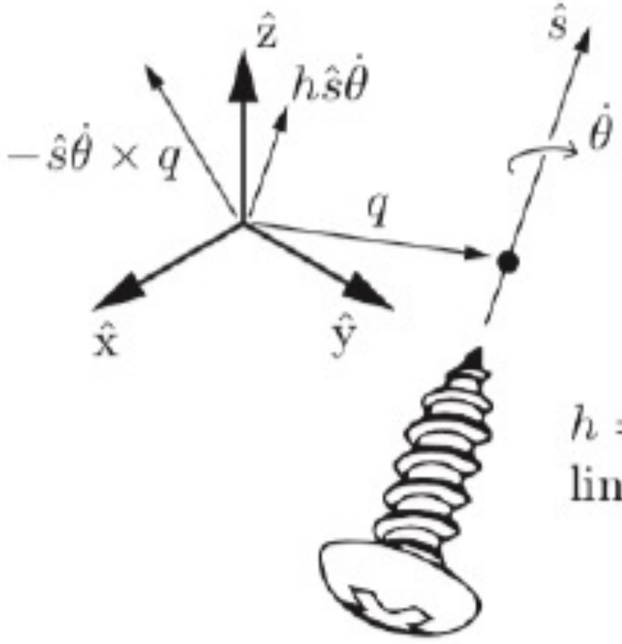
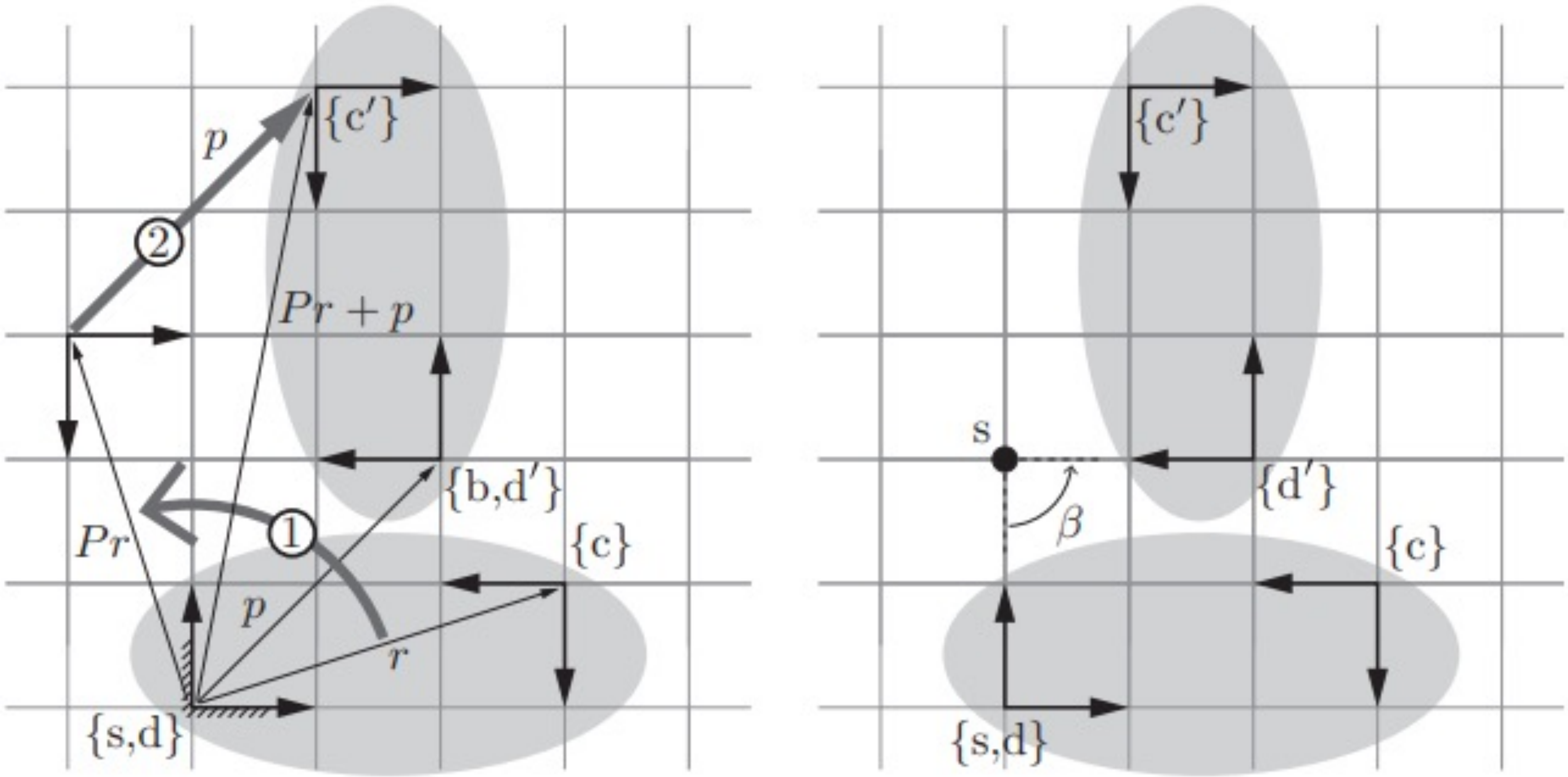
3 Screw motions give us a generalized concept for defining any motion with a direction and a "screw".
From the information in a twist, we can compute all the components for a screw motion.
Can we connect screws with exponential coordinates to give us a general equation for T?

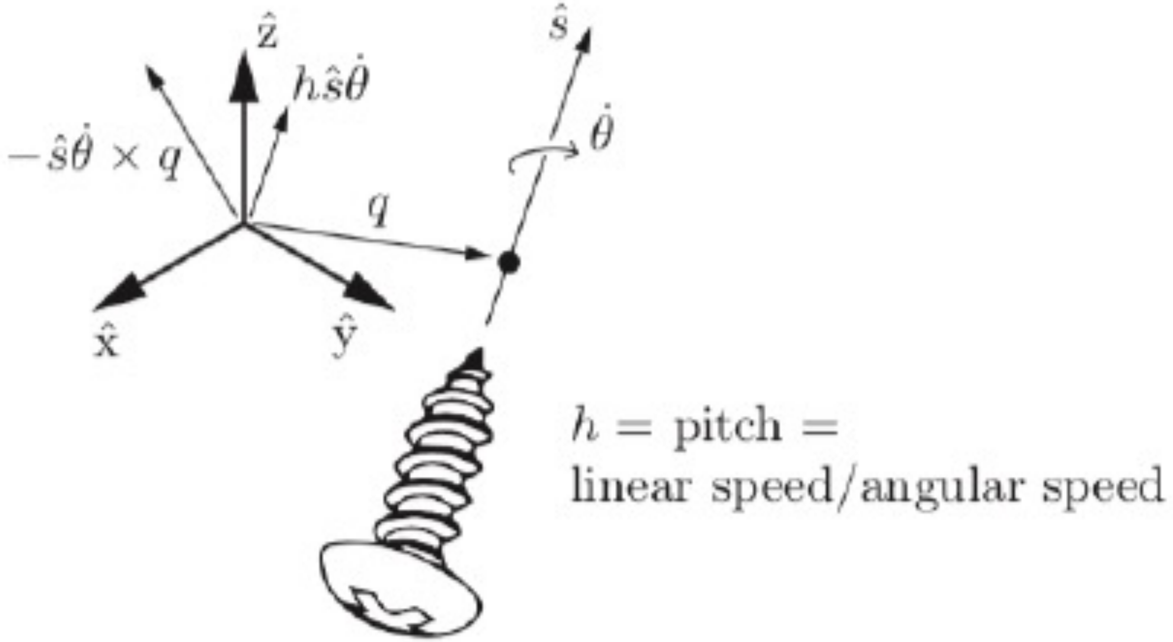


Screw Motions



Screw motions in 3D

Chasles-Mozzi Theorem:
Any displacement in 3D can be represented by a rotation and translation about the same axis, referred to as a screw motion



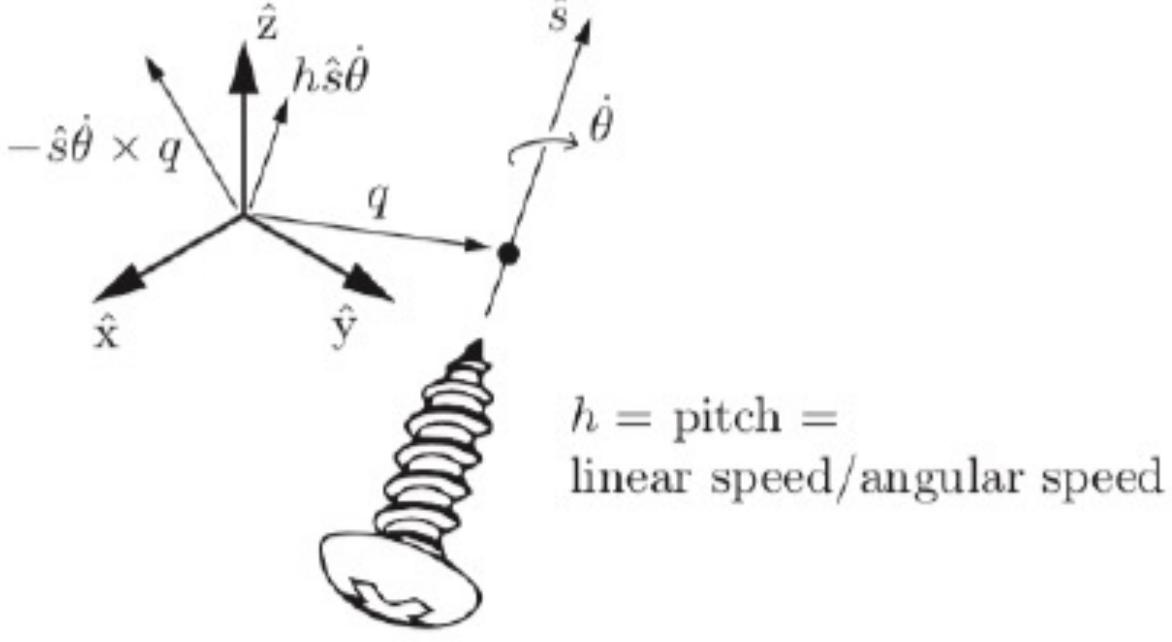
$q \in \mathbb{R}^3$ is any point along the axis

$\hat{s}$ is a unit vector in the direction of the axis

h is the **screw pitch**, which is the ratio between the linear and angular speed along axis

We write this as $\mathcal{S} = \{q, \hat{s}, h\}$

Screw motions and Twists



Screw axis

Exponential Coordinates for Rigid Body Motions

- Recall for rotation matrices: $R = e^{[\hat{\omega}]\theta}$
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  $\mathcal{S}\theta \in \mathbb{R}^6$, where $\mathcal{S}$ is the screw axis and θ is the distance about $\mathcal{S}$ to take frame from I to T

$$T = e^{[\mathcal{S}]\theta} = I + [\mathcal{S}]\theta + [\mathcal{S}]^2 \frac{\theta^2}{2!} + \cdots$$

$$= \begin{bmatrix} e^{[\hat{\omega}]\theta} & G(\theta)v \\ 0 & 1 \end{bmatrix}$$

$$G(\theta) = I\theta + [\omega]\frac{\theta^2}{2!} + \cdots = I\theta + (1 - \cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2$$

What does the screw axis capture?

- A twist for 1 second around the screw axis at angular velocity θ, then the resulting frame is T
- A twist for θ seconds around the screw axis at angular velocity 1, then the resulting frame is T

Example of Screw Motion for Transformations

Find the pose that is produced by a given screw

Frame 0 is fixed in space and frame 1 is fixed in a body. The current pose of frame 1 in the coordinates of frame 0 is M. The body undergoes a body screw motion with axis $\mathcal{B}$ and magnitude θ. Find the new pose T_{01} of frame 1 in the coordinates of frame 0.

matlab python

```
M = [0.03262475 -0.39471442 0.91822445 -0.58853212; -0.21629944 -0.8997297
B = [0.00000000; 0.00000000; 0.00000000; -0.67130813; -0.02965809; 0.74058
theta = 1.29638080;
```

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matlab python

```
M = [0.03262475 -0.3
B = [0.00000000; 0.0
theta = 1.29638080;
```

copy this text

Note that my code snippet is in Julia!

[1]:
```
M = [0.03262475 -0.39471442 0.91822445 -0.58853212; -0.21629944 -0.89972977 -0.37907900
B = [0.00000000; 0.00000000; 0.00000000; -0.67130813; -0.02965809; 0.74058477];
theta = 1.29638080;

bb=[0 -B[3] B[2] B[4];
    B[3] 0 -B[1] B[5];
    -B[2] B[1] 0 B[6];
    0 0 0 0];
tt=exp(bb*theta);
T=M*tt
```

[1]:
```
4×4 Array{Float64,2}:
  0.0326248  -0.394714   0.918224   0.27982
 -0.216299   -0.89973   -0.379079   0.61705
  0.975782   -0.186244  -0.11473   -0.285795
  0.0         0.0        0.0        1.0
```

How to compute?

Proposition 3.25. *Let $\mathcal{S} = (\omega, v)$ be a screw axis. If $\|\omega\| = 1$ then, for any distance $\theta \in \mathbb{R}$ traveled along the axis,*

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} e^{[\omega]\theta} & \left(I\theta + (1-\cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2\right) v \\ 0 & 1 \end{bmatrix}. \tag{3.88}$$

If $\omega = 0$ and $\|v\| = 1$, then

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix}.$$

Compute $e^{[\omega]\theta}$ just like any matrix exponential!

$$e^{[\omega]\theta} = I + [\omega]\theta + [\omega]^2\frac{\theta^2}{2!} + [\omega]^3\frac{\theta^3}{3!} + \cdots$$

$$e^{[\omega]\theta} = I + \sin\theta\,[\omega] + (1-\cos\theta)[\omega]^2$$

Logarithm of rigid body motions

Given $T(R, p) \in SE(3)$, find $\mathcal{S} = [\omega \quad v]^\top$ and θ such that:

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \rightarrow [\mathcal{S}]\theta = \begin{bmatrix} [\omega]\theta & v\theta \\ 0 & 0 \end{bmatrix}$$

Logarithm of rigid body motions

Given $T(R, p) \in SE(3)$, find $\mathcal{S} = [\omega \quad v]^{\mathsf{T}}$ and θ such that:

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Define the logarithm of T

1. If $R = I$, then $\omega = 0$ and $v = \frac{p}{\|p\|}$ and $\theta = \|p\|$
2. Find ω, θ using matrix logarithm of R (see earlier slide!)
   Then, $v = G^{-1}(\dot{\theta})p$, where $G^{-1}(\dot{\theta}) = \frac{1}{\theta} I - \frac{1}{2}[\omega] + \left(\frac{1}{\theta} - \frac{1}{2}\cot\left(\frac{\theta}{2}\right)\right)[\omega]^2$

Now we can go back and forth between exponential coordinates and transformations!

Summary

- Used transformations to define spatial velocities as **body and spatial twists**
- Introduced **screw motions** and the screw interpretation of a twist
- The **screw axis $\mathcal{S}$** was defined and used to define the **exponential coordinates of homogeneous transformations**
- Connections between last few lectures:
  - Transformation matrix T is analogous to rotation matrix R
  - A screw axis $\mathcal{S}$ is analogous to rotation axis $\widehat{\omega}$
  - A twist $\mathcal{V}$ can be expressed as $\mathcal{S}\dot{\theta}$ and is analogous to angular velocity $\omega = -\widehat{\omega}\dot{\theta}$
  - Exponential coordinates $\mathcal{S}\theta \in \mathbb{R}^6$ for rigid body motions are analogous to exponential coordinates $\widehat{\omega}\theta \in \mathbb{R}^3$