

Lecture 06: Screw Theory

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Modern Robotics Chapters 3.3-3.4

Quick Recap of Past Lectures

- 1 We can define rotation matrices with a matrix exponential: $R = e^{[\hat{\omega}]\theta}$
Where $e^{[\hat{\omega}]\theta}$ is computed with Rodrigues's Formula
 $\hat{\omega}\theta$ are referred to as exponential coordinates

Quick Recap of Past Lectures

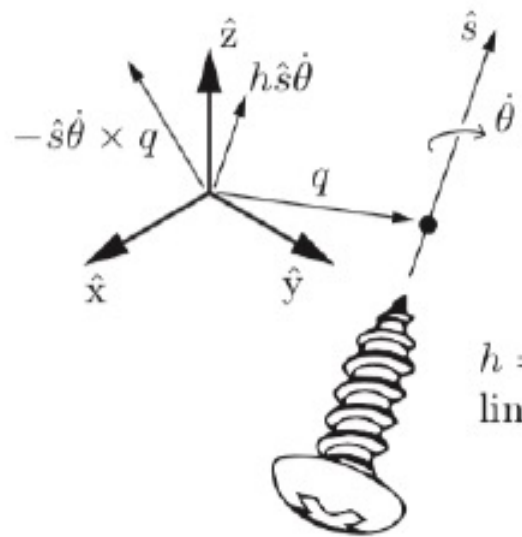
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For transformations (rotations + translations), we need angular and linear (spatial) velocities!
Twists are the concatenation of ω and v

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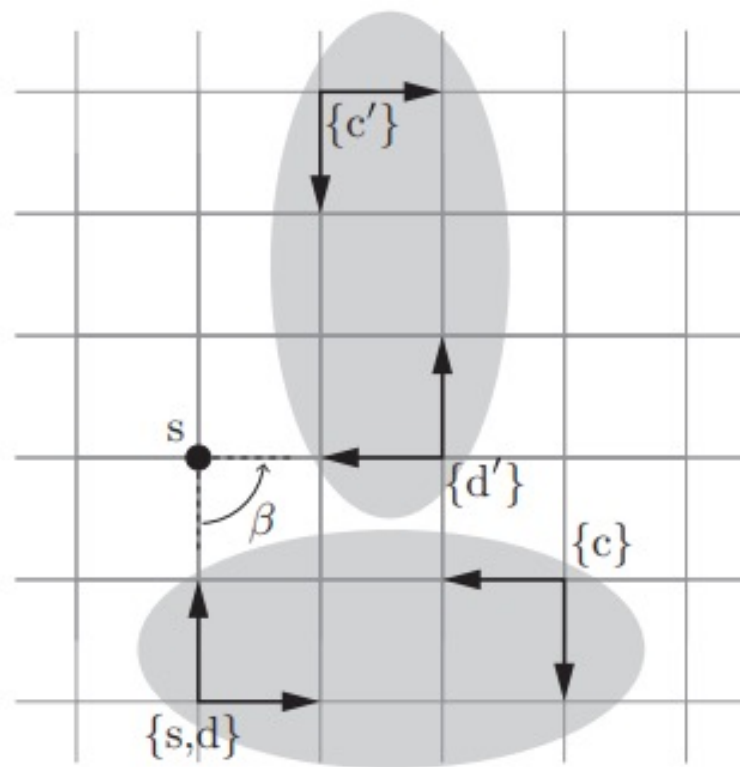
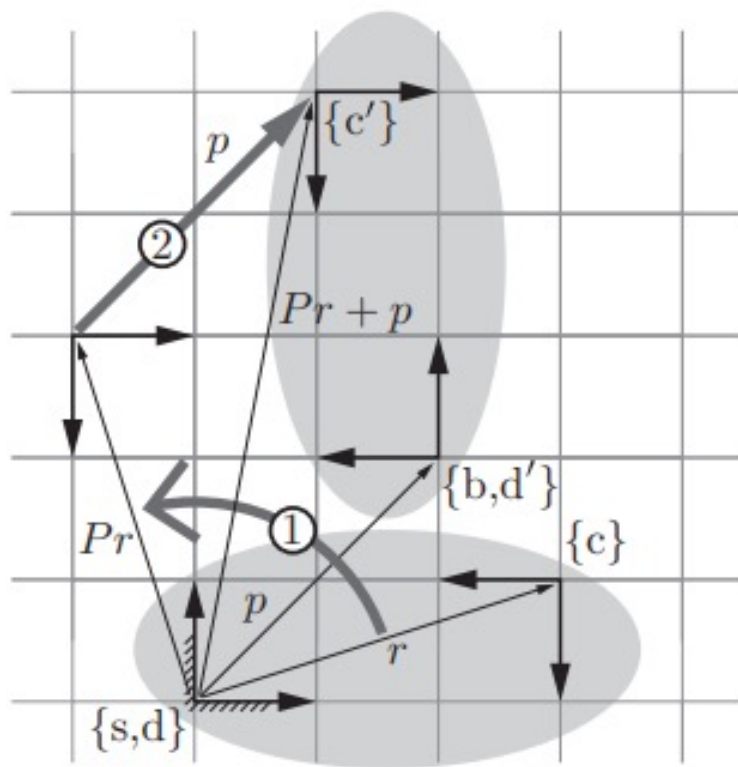
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3 Screw motions give us a generalized concept for defining any motion with a direction and a "screw".
From the information in a twist, we can compute all the components for a screw motion.
Can we connect screws with exponential coordinates to give us a general equation for T ?



Screw Motions



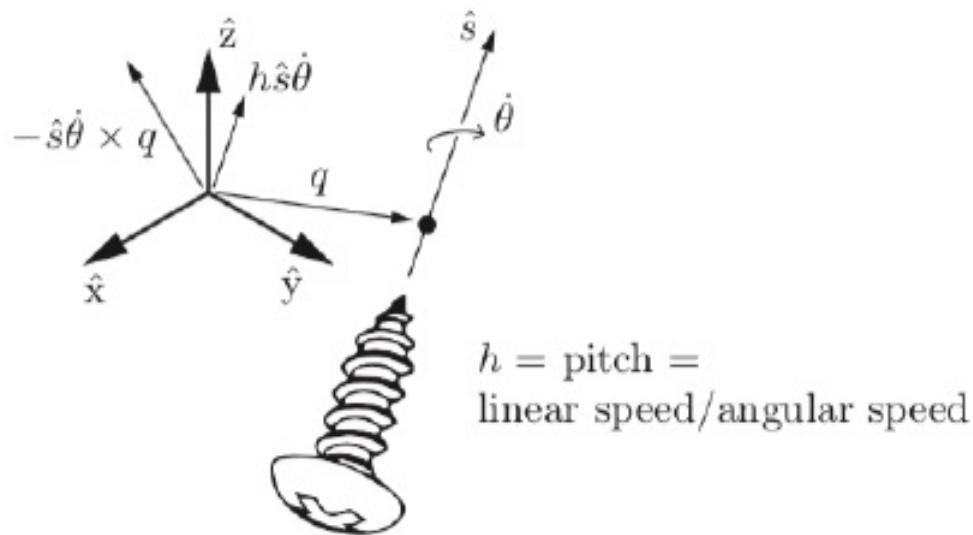
Screw motions in 3D

Chasles-Mozzi Theorem:

Any displacement in 3D can be represented by a rotation and translation about the same axis, referred to as a screw motion

- $q \in \mathbb{R}^3$ is any point along the axis
- $\hat{s}$ is a unit vector in the direction of the axis
- h is the **screw pitch**, which is the ratio between the linear and angular speed along axis

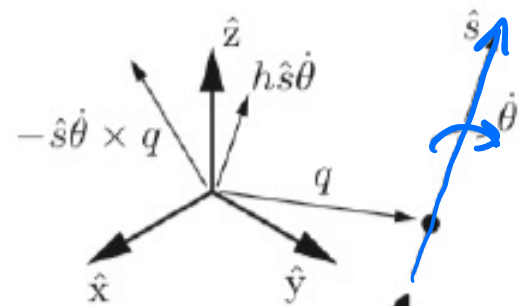
We write this as $\mathcal{S} = \{q, \hat{s}, h\}$



Screw motions and Twists

screw to twist:

$$v = \begin{bmatrix} \omega \\ v \end{bmatrix} = \begin{bmatrix} \hat{s}\dot{\theta} \\ h\dot{\theta} - \hat{s}\dot{\theta} \times q \end{bmatrix}$$



$h = \text{pitch} =$
linear speed/angular speed

twist to screw:

$$\hat{s} = \omega / \|\omega\| = \hat{\omega}, \quad \dot{\theta} = \|\omega\|, \quad h = \frac{\hat{\omega}^T v}{\dot{\theta}} = \omega^T v / \|\omega\|^2$$

assuming $\omega \neq 0$, find q s.t.

$$q = \frac{\hat{s} \times v}{\dot{\theta}}$$

Screw axis

$$S = \begin{bmatrix} \omega \\ v \end{bmatrix} \in \mathbb{R}^6 \quad | \quad \text{note: these are overloaded variables}$$

where either:

① $\|\omega\| = 1$

OR ② $\omega = 0$ and $\|v\| = 1$

$$v = -\omega \times q + \underbrace{h\omega}_{h=0}$$

→ pure rot

→ translating along v

$$[S] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \in \mathfrak{se}(3), \quad S_a = [Ad_{T_{ab}} S_b]$$

Exponential Coordinates for Rigid Body Motions

- Recall for rotation matrices: $R = e^{[\hat{\omega}]\theta}$
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Exponential Coordinates for Rigid Body Motions

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- What are exponential coordinates for T ?

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$$T = e^{[\mathcal{S}]\theta} = I + [\mathcal{S}]\theta + [\mathcal{S}]^2 \frac{\theta^2}{2!} + \dots$$

$$= \begin{bmatrix} e^{[\hat{\omega}]\theta} & G(\theta)v \\ 0 & 1 \end{bmatrix}$$

$$G(\theta) = I\theta + [\omega] \frac{\theta^2}{2!} + \dots = I\theta + (1 - \cos \theta)[\omega] + (\theta - \sin \theta)[\omega]^2$$

What does the screw axis capture?

- A twist for 1 second around the screw axis at angular velocity θ , then the resulting frame is T
- A twist for θ seconds around the screw axis at angular velocity 1, then the resulting frame is T

Example of Screw Motion for Transformations

Find the pose that is produced by a given screw

Frame 0 is fixed in space and frame 1 is fixed in a body. The current pose of frame 1 in the coordinates of frame 0 is M . The body undergoes a body screw motion with axis B and magnitude θ . Find the new pose T_{01} of frame 1 in the coordinates of frame 0.

matlab

python

```
M = [0.03262475 -0.39471442 0.91822445 -0.58853212; -0.21629944 -0.8997297  
B = [0.00000000; 0.00000000; 0.00000000; -0.67130813; -0.02965809; 0.74058  
theta = 1.29638080;
```

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Example of Screw Motion for Transformations

Find the pose that is produced by a given screw

Note that my code snippet is in Julia!

Frame 0 is fixed in space and frame 1 is fixed in a body. The current pose of frame 1 in the coordinates of frame 0 is M . The body undergoes a body screw motion with axis B and magnitude θ . Find the new pose T_{01} of frame 1 in the coordinates of frame 0.

matlab

python

```
M = [0.03262475 -0.39471442 0.91822445 -0.58853212; -0.21629944 -0.89972977 -0.37907900 0.975782; 0.00000000 0.00000000 0.00000000 -0.67130813; -0.02965809 0.74058477 0.00000000 0.00000000];
B = [0.00000000; 0.00000000; 0.00000000; -0.67130813; -0.02965809; 0.74058477];
theta = 1.29638080;
```

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```
[1]: M = [0.03262475 -0.39471442 0.91822445 -0.58853212; -0.21629944 -0.89972977 -0.37907900 0.975782; 0.00000000 0.00000000 0.00000000 -0.67130813; -0.02965809 0.74058477 0.00000000 0.00000000];
B = [0.00000000; 0.00000000; 0.00000000; -0.67130813; -0.02965809; 0.74058477];
theta = 1.29638080;
```

```
bb= [ 0 -B[3] B[2] B[4]
      B[3] 0 -B[1] B[5]
      -B[2] B[1] 0 B[6]
      0 0 0 0];
```

```
tt=exp(bb*theta);
```

```
T=M*tt
```

$\leftarrow [B]$
 $\leftarrow$ transformation assoc w/ B

```
[1]: 4x4 Array{Float64,2}:
 0.0326248 -0.394714  0.918224  0.27982
-0.216299 -0.89973  -0.379079  0.61705
 0.975782 -0.186244 -0.11473  -0.285795
 0.0       0.0       0.0       1.0
```

How to compute?

Proposition 3.25. *Let $\mathcal{S} = (\omega, v)$ be a screw axis. If $\|\omega\| = 1$ then, for any distance $\theta \in \mathbb{R}$ traveled along the axis,*

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} e^{[\omega]\theta} & (I\theta + (1 - \cos \theta)[\omega] + (\theta - \sin \theta)[\omega]^2) v \\ 0 & 1 \end{bmatrix}. \quad (3.88)$$

If $\omega = 0$ and $\|v\| = 1$, then

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix}.$$

Compute $e^{[\omega]\theta}$ just like any matrix exponential!

$$e^{[\omega]\theta} = I + [\omega]\theta + [\omega]^2 \frac{\theta^2}{2!} + [\omega]^3 \frac{\theta^3}{3!} + \dots$$

$$e^{[\omega]\theta} = I + \sin \theta [\omega] + (1 - \cos \theta)[\omega]^2$$

Logarithm of rigid body motions

Given $T(R, p) \in SE(3)$, find $\mathcal{S} = [\omega \quad v]^\top$ and θ such that:

$$T = e^{[\mathcal{S}]\theta} = \underbrace{\begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}} \rightarrow \underbrace{[\mathcal{S}]\theta}_{\text{arrow from } \mathcal{S}} = \begin{bmatrix} [\omega]\theta & v\theta \\ 0 & 0 \end{bmatrix}$$

Logarithm of rigid body motions

Given $T(R, p) \in SE(3)$, find $\mathcal{S} = [\omega \quad v]^\top$ and θ such that:

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \rightarrow [\mathcal{S}]\theta = \begin{bmatrix} [\omega]\theta & v\theta \\ 0 & 0 \end{bmatrix}$$

Define the logarithm of T

1. If $R = I$, then $\omega = 0$ and $v = \frac{p}{\|p\|}$ and $\theta = \|p\|$
2. Find ω, θ using matrix logarithm of R (see earlier slide!)
Then, $v = G^{-1}(\dot{\theta})p$, where $G^{-1}(\dot{\theta}) = \frac{1}{\theta}I - \frac{1}{2}[\omega] + \left(\frac{1}{\theta} - \frac{1}{2}\cot\left(\frac{\theta}{2}\right)\right)[\omega]^2$

Now we can go back and forth between exponential coordinates and transformations!

Summary

- Used transformations to define spatial velocities as **body and spatial twists**
- Introduced **screw motions** and the screw interpretation of a twist
- The **screw axis $\mathcal{S}$** was defined and used to define the **exponential coordinates of homogeneous transformations**
- Connections between last few lectures:
 - Transformation matrix T is analogous to rotation matrix R
 - A screw axis $\mathcal{S}$ is analogous to rotation axis $\hat{\omega}$
 - A twist $\mathcal{V}$ can be expressed as $\mathcal{S}\dot{\theta}$ and is analogous to angular velocity $\omega = -\hat{\omega}\dot{\theta}$
 - Exponential coordinates $\mathcal{S}\theta \in \mathbb{R}^6$ for rigid body motions are analogous to exponential coordinates $\hat{\omega}\theta \in \mathbb{R}^3$