Lecture 05: twists, screws, and wrenches

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Modern Robotics Chapters 3.3-3.4

Administrivia

- Note that for those who took TAM courses, we use a different frame of reference for rotations / transformations!
- Office hours are now posted
  - If you email me for an appointment, please put some combination of ECE 470 / ME 445 / AE 482 clearly in the subject line
- Exam starts next week
  - Content from this week will be covered on the exam
  - Topics and questions will be **very similar** to the homeworks – those will be your best practices
  - Please do the CBTF overview quiz

Who is Michael Chasles?

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- The proof that a spatial displacement can be decomposed into a rotation and slide around and along a line is attributed to the astronomer and mathematician Giulio Mozzi (1763),
  - In Italy, the screw axis is traditionally called *asse di Mozzi*
- However, most textbooks refer to a subsequent similar work by Michel Chasles dating 1830
- Several other scholars/contemporaries of M. Chasles obtained the same or similar results around that time, including G. Giorgini, Cauchy, Poinsot, Poisson, and Rodrigues

Recap from last few lectures

Rotations

Transformations

Recall the cross product

- The cross-product is defined as
$$a{\times}b = \|a\|\|b\| \sin\theta \,\hat{n}$$

- In coordinates:
$$a{\times}b = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}$$

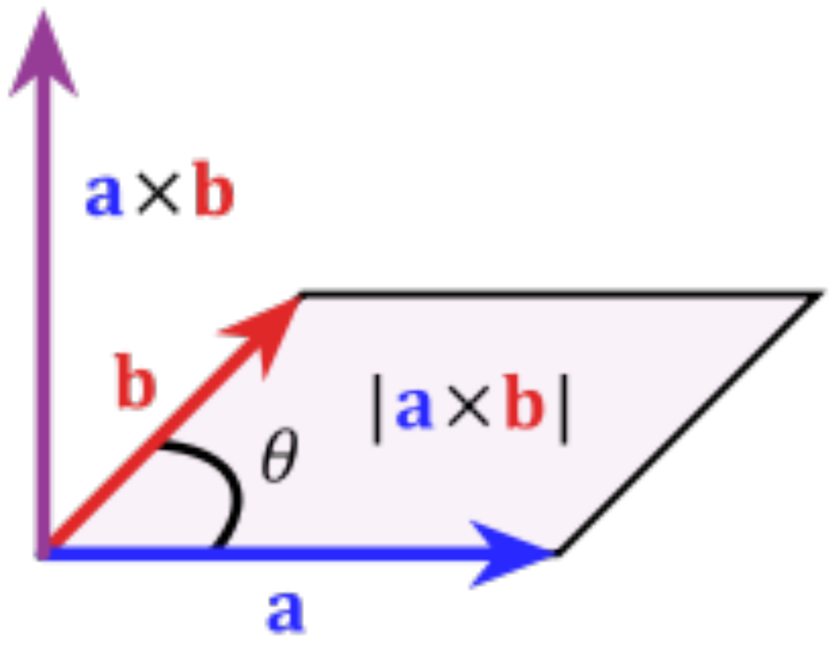


Image Credit: Wikipedia

Exponential Coordinate Representation

- Instead representing orientation as a rotation matrix, we introduce a three-parameter representation: **exponential coordinates**
- Recall: $\hat{\omega}$ rotation axis, θ angle of rotation
- Then $\hat{\omega}\theta \in \mathbb{R}^3$gives the exponential coordinate representation

- A new interpretation for a frame coincident with $\{s\}$:
  - A rotation for 1 second around $\hat{\omega}$ at angular velocity θ, then the resulting frame is R
  - A rotation for θ seconds around $\hat{\omega}$ at angular velocity 1, then the resulting frame is R

Recall Exponential Coordinates for Rotations

T(t) describes the pose of a body

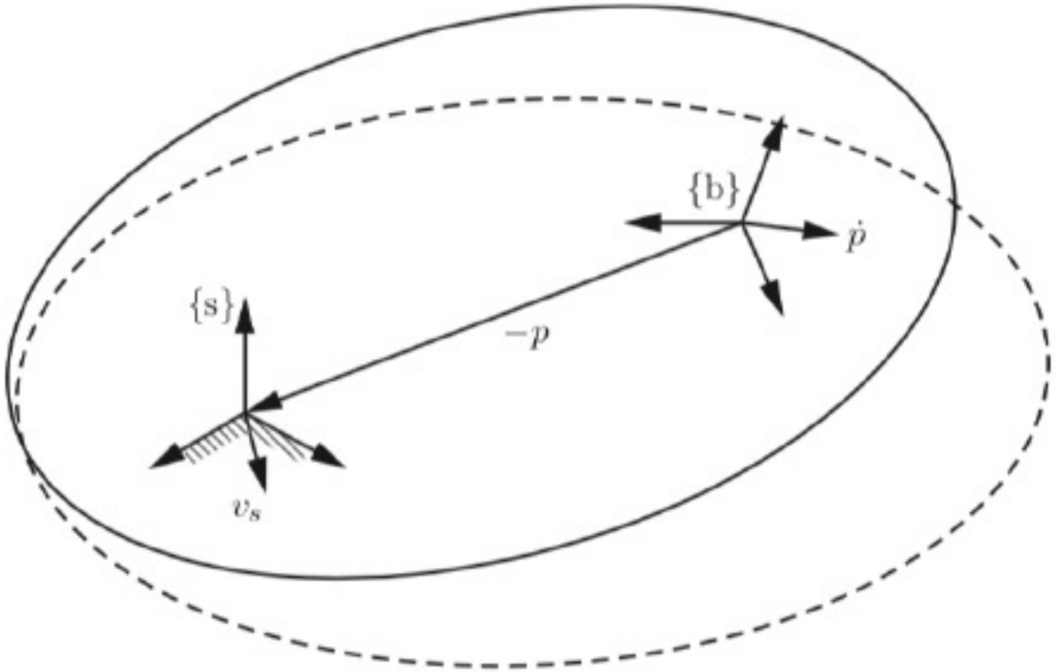
Moving Frames: linear and angular velocity (1)

Moving Frames: linear and angular velocity (2)

Linear and angular velocity in reference frame

Physical interpretation of v_s

- Assume that a very large rigid body is attached to the frame {b}, and is large enough to contain the origin of {s}
- What is the velocity of the point of this body as the origin of {s}?
- There are two components:
  - $\dot{p}$: the motion of the body
  - $\omega_s \times (-p)$: the rotation of the body



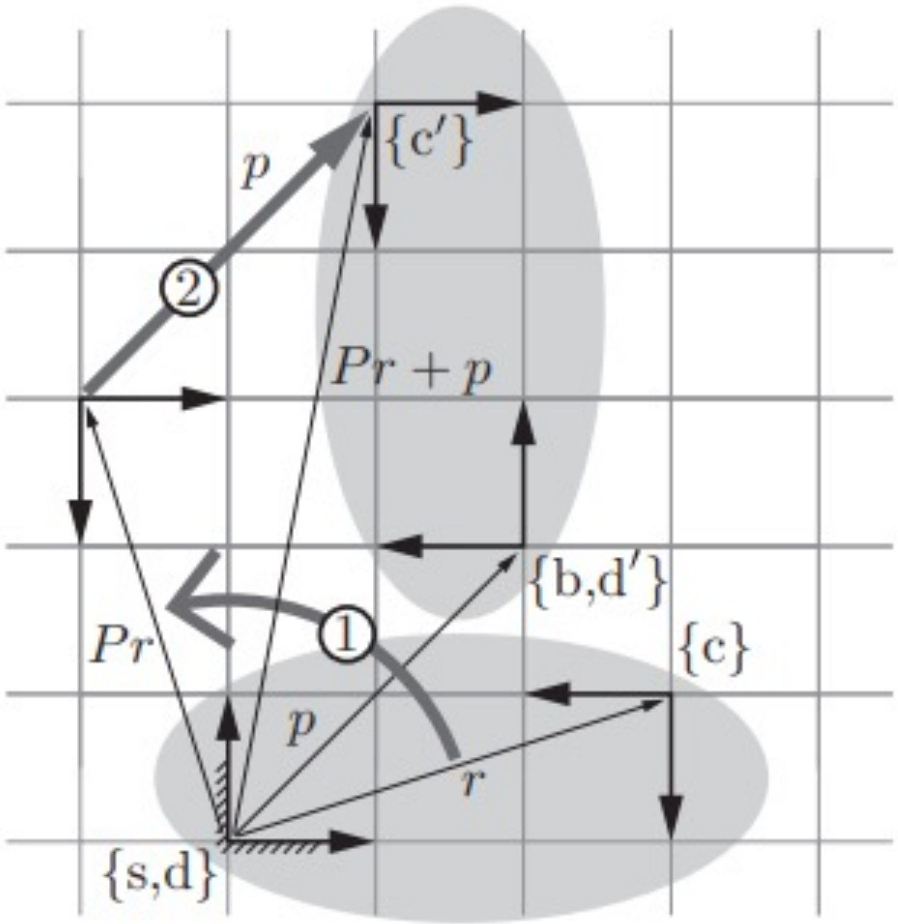
Twists

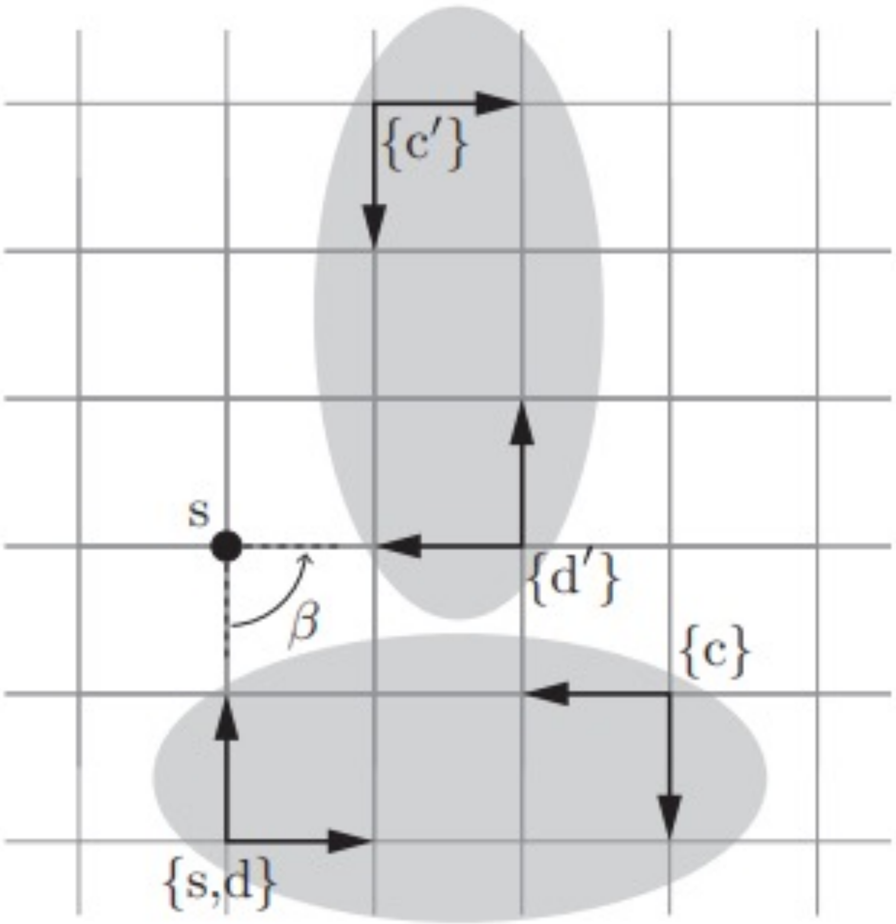
Adjoint Map associated with T: $Ad_T(\mathcal{V})$

$$\mathcal{V}' = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \mathcal{V} = [Ad_T]\mathcal{V}$$

- By construction, $\mathcal{V}' = Ad_T(\mathcal{V}) \Leftrightarrow [\mathcal{V}'] = T[\mathcal{V}]T^{-1}$
- Properties:
  1. $Ad_{T_1}\left(Ad_{T_2}(\mathcal{V})\right) = \left[Ad_{T_1}\right]\left[Ad_{T_2}\right]\mathcal{V} = Ad_{T_1 T_2}(\mathcal{V})$
  2. $\left[Ad_{T_1^{-1}}\right] = \left[Ad_{T_1}\right]^{-1}$

Screw Motions





2D Screw Motions

- Any rigid motion in the plane can be represented by a rotation around a well-chosen center
- We can encode it with (β, s_x, s_y), where (s_x, s_y) is the position of the center of the rotation, and β the angle

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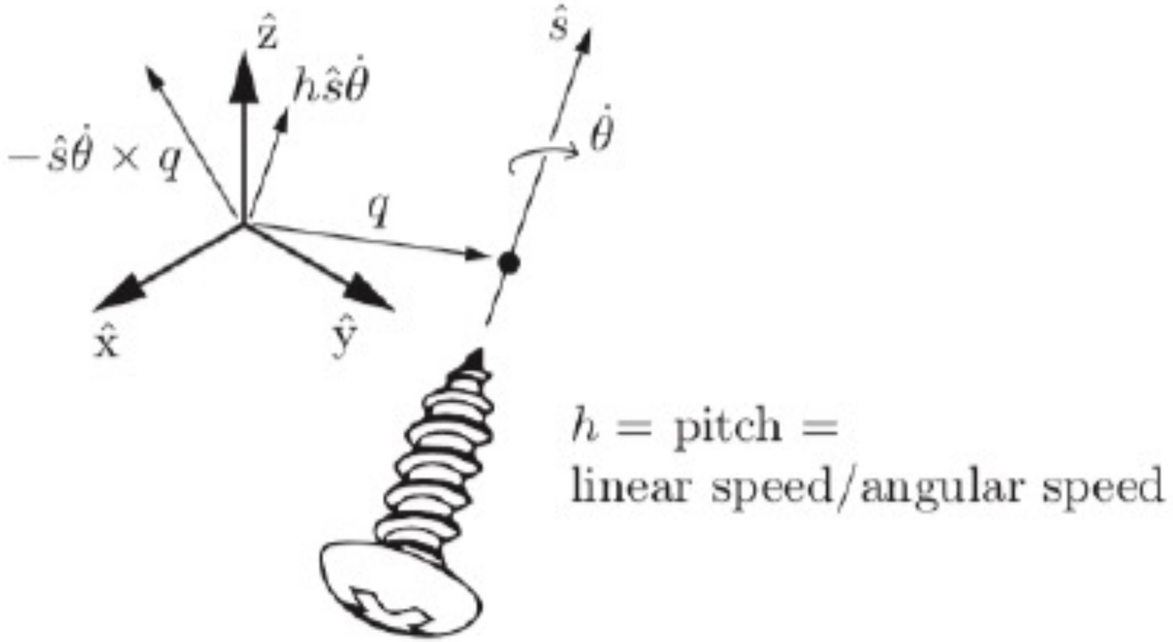
- Recall that the angular velocity ω can be viewed as $\widehat{\omega}\dot{\theta}$, where $\widehat{\omega}$ is the unit rotation axis and $\dot{\theta}$ is the rate of rotation
- A twist $\mathcal{V}$ can be interpreted in terms of a screw axis $\mathcal{S}$ and a velocity $\dot{\theta}$ about that axis

Screw motions in 3D

Chasles-Mozzi Theorem:
Any displacement in 3D can be represented by a rotation and translation about the same axis, referred to as a screw motion



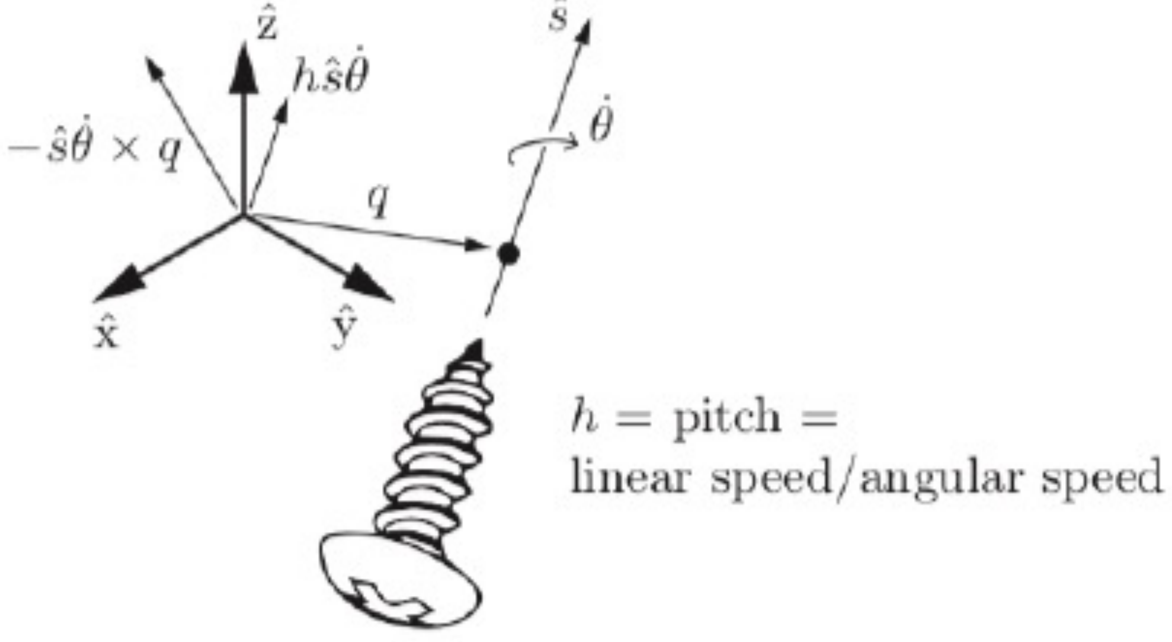
$q \in \mathbb{R}^3$ is any point along the axis

$\hat{s}$ is a unit vector in the direction of the axis

h is the **screw pitch**, which is the ratio between the linear and angular speed along axis

We write this as $\mathcal{S} = \{q, \hat{s}, h\}$

Screw motions and Twists



Screw axis

Exponential Coordinates for Rigid Body Motions

What does the screw axis capture?

- A twist for 1 second around the screw axis at angular velocity θ, then the resulting frame is T
- A twist for θ seconds around the screw axis at angular velocity 1, then the resulting frame is T

Example of Screw Motion for Transformations

Find the pose that is produced by a given screw

Frame 0 is fixed in space and frame 1 is fixed in a body. The current pose of frame 1 in the coordinates of frame 0 is M. The body undergoes a body screw motion with axis $\mathcal{B}$ and magnitude θ. Find the new pose T_{01} of frame 1 in the coordinates of frame 0.

matlab python

```
M = [0.03262475 -0.39471442 0.91822445 -0.58853212; -0.21629944 -0.8997297
B = [0.00000000; 0.00000000; 0.00000000; -0.67130813; -0.02965809; 0.74058
theta = 1.29638080;
```

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Example of Screw Motion for Transformations

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matlab python

```
M = [0.03262475 -0.39
B = [0.00000000; 0.00
theta = 1.29638080;
```

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Note that my code snippet is in Julia!

[1]:
```
M = [0.03262475 -0.39471442 0.91822445 -0.58853212; -0.21629944 -0.89972977 -0.37907900
B = [0.00000000; 0.00000000; 0.00000000; -0.67130813; -0.02965809; 0.74058477];
theta = 1.29638080;

bb=[0 -B[3] B[2] B[4];
    B[3] 0 -B[1] B[5];
    -B[2] B[1] 0 B[6];
    0 0 0 0];
tt=exp(bb*theta);
T=M*tt
```

[1]:
```
4×4 Array{Float64,2}:
  0.0326248  -0.394714   0.918224   0.27982
 -0.216299   -0.89973   -0.379079   0.61705
  0.975782   -0.186244  -0.11473   -0.285795
  0.0         0.0        0.0        1.0
```

How to compute?

Proposition 3.25. *Let $\mathcal{S} = (\omega, v)$ be a screw axis. If $\|\omega\| = 1$ then, for any distance $\theta \in \mathbb{R}$ traveled along the axis,*

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} e^{[\omega]\theta} & \left(I\theta + (1-\cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2\right) v \\ 0 & 1 \end{bmatrix}. \tag{3.88}$$

If $\omega = 0$ and $\|v\| = 1$, then

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix}.$$

Compute $e^{[\omega]\theta}$ just like any matrix exponential!

$$e^{[\omega]\theta} = I + [\omega]\theta + [\omega]^2 \frac{\theta^2}{2!} + [\omega]^3 \frac{\theta^3}{3!} + \cdots$$

$$e^{[\omega]\theta} = I + \sin\theta\,[\omega] + (1-\cos\theta)[\omega]^2$$

Logarithm of rigid body motions

Given $T(R, p) \in SE(3)$, find $\mathcal{S} = [\omega \quad v]^{\mathsf{T}}$ and θ such that:

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \rightarrow [\mathcal{S}]\theta = \begin{bmatrix} [\omega]\theta & v\theta \\ 0 & 0 \end{bmatrix}$$

Define the logarithm of T

1. If $R = I$, then $\omega = 0$ and $v = \frac{p}{\|p\|}$ and $\theta = \|p\|$
2. Find ω, θ using matrix logarithm of R (see earlier slide!)
   Then, $v = G^{-1}(\dot{\theta})p$, where $G^{-1}(\dot{\theta}) = \frac{1}{\theta}I - \frac{1}{2}[\omega] + \left(\frac{1}{\theta} - \frac{1}{2}\cot\left(\frac{\theta}{2}\right)\right)[\omega]^2$

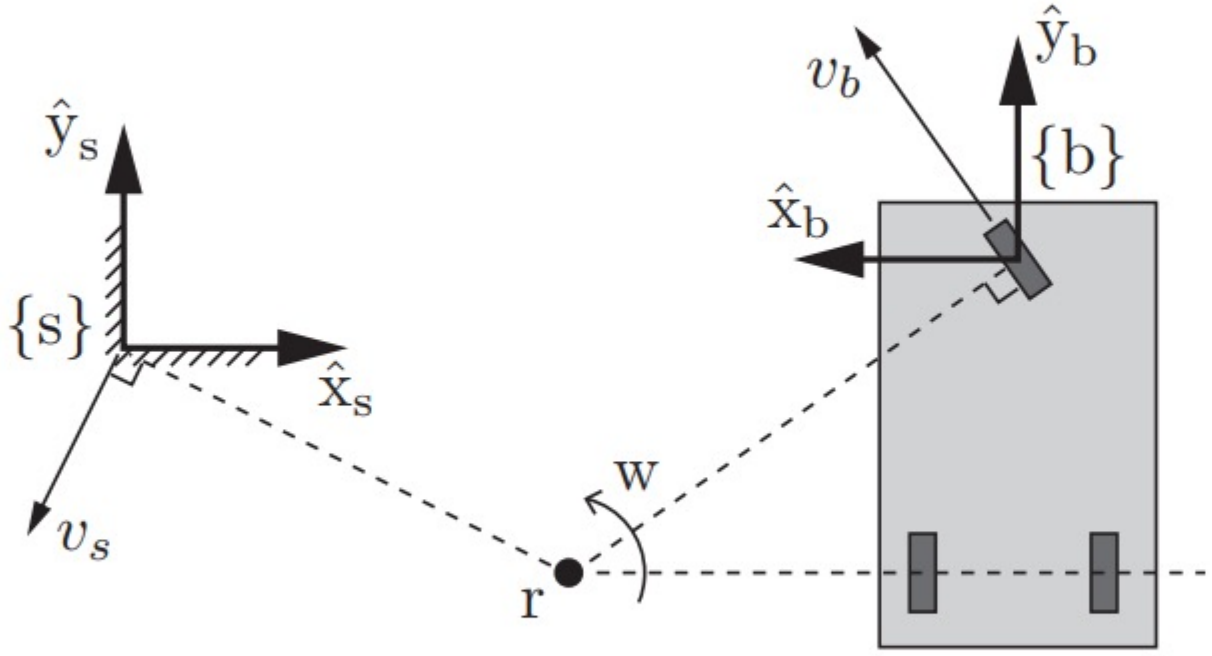
Now we can go back and forth between exponential coordinates and transformations!

Summary

- Used transformations to define spatial velocities as **body and spatial twists**
- Introduced **screw motions** and the screw interpretation of a twist
- The **screw axis $\mathcal{S}$** was defined and used to define the **exponential coordinates of homogeneous transformations**
- Connections between last lecture and this lecture:
  - Transformation matrix T is analogous to rotation matrix R
  - A screw axis $\mathcal{S}$ is analogous to rotation axis $\widehat{\omega}$
  - A twist $\mathcal{V}$ can be expressed as $\mathcal{S}\dot{\theta}$ and is analogous to angular velocity $\omega = -\widehat{\omega}\dot{\theta}$
  - Exponential coordinates $\mathcal{S}\theta \in \mathbb{R}^6$ for rigid body motions are analogous to exponential coordinates $\widehat{\omega}\theta \in \mathbb{R}^3$

Extra Slides

Twist Visualization



The twist corresponding to the instantaneous motion of the chassis of a three-wheeled vehicle can be visualized as an angular velocity w about the point r.