Lecture 05: twists, screws, and wrenches

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February 3, 2026

Modern Robotics Chapters 3.3-3.4

Administrivia

- Note that for those who took TAM courses, we use a different frame of reference for rotations / transformations!
- Office hours are now posted
  - If you email me for an appointment, please put some combination of ECE 470 / ME 445 / AE 482 clearly in the subject line
- Exam starts next week
  - Content from this week will be covered on the exam
  - Topics and questions will be **very similar** to the homeworks – those will be your best practices
  - Please do the CBTF overview quiz

Who is Michael Chasles?

- Rodrigues and Chasles took the entrance exam to Polytechnique/Normale at the same time, finishing first and second respectively
  - Rodrigues did not use it and elected to go to La Sorbonne

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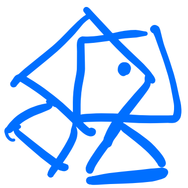
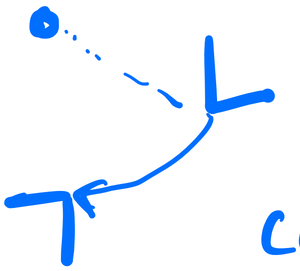
- Rodrigues and Chasles took the entrance exam to Polytechnique/Normale at the same time, finishing first and second respectively
  - Rodrigues did not use it and elected to go to La Sorbonne
- The proof that a spatial displacement can be decomposed into a rotation and slide around and along a line is attributed to the astronomer and mathematician Giulio Mozzi (1763),
  - In Italy, the screw axis is traditionally called *asse di Mozzi*
- However, most textbooks refer to a subsequent similar work by Michel Chasles dating 1830
- Several other scholars/contemporaries of M. Chasles obtained the same or similar results around that time, including G. Giorgini, Cauchy, Poinsot, Poisson, and Rodrigues

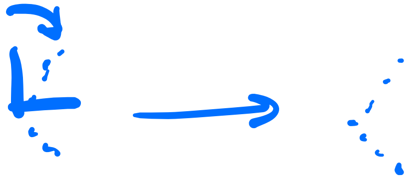
Recap from last few lectures

Rotations





Transformations



$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$$

can represnt: · pose (ori, pos)

· operation

Recall the cross product

- The cross-product is defined as

$$a \times b = \|a\|\|b\| \sin\theta\, \hat{n}$$

- In coordinates:

$$a \times b = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

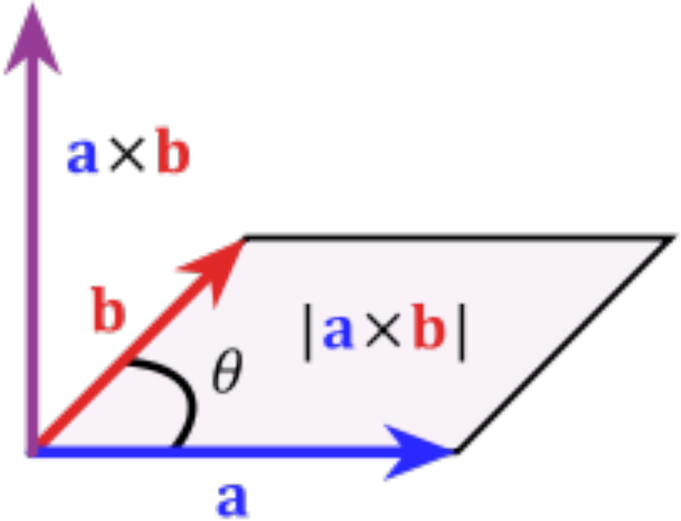


Image Credit: Wikipedia

Exponential Coordinate Representation

- Instead representing orientation as a rotation matrix, we introduce a three-parameter representation: **exponential coordinates**
- Recall: $\widehat{\omega}$ rotation axis, θ angle of rotation
- Then $\widehat{\omega}\theta \in \mathbb{R}^3$gives the exponential coordinate representation

- A new interpretation for a frame coincident with $\{s\}$:
  - A rotation for 1 second around $\widehat{\omega}$ at angular velocity θ, then the resulting frame is R
  - A rotation for θ seconds around $\widehat{\omega}$ at angular velocity 1, then the resulting frame is R

Recall Exponential Coordinates for Rotations

$$\dot{R}(t) = \omega \times R = [\omega] R$$

$$\dot{x}(t) = A x(t)$$

$$[\omega] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \in so(3)$$

$$A = -A^T$$

$$R(\hat{\omega}, \theta) = e^{[\hat{\omega}]\theta} = \underbrace{I + \sin\theta\,[\hat{\omega}] + (1-\cos\theta)[\hat{\omega}]^2}_{\text{Rodrigues's Formula}}$$

T(t) describes the pose of a body

orientation

position / translation

$$T = \begin{bmatrix} R^{3\times 3} & p^{3\times 1} \\ 0^{1\times 3} & 1^{1\times 1} \end{bmatrix} \in SE(3)$$

homogeneous rep

$$R = [\hat{x} \ \ \hat{y} \ \ \hat{z}]$$

recall: $\dot{R} = [\omega] R \rightsquigarrow [\omega_s] = \dot{R} R^T$

want: $\dot{T} = [?] T \rightsquigarrow [?] = \dot{T} T^{-1}$

Moving Frames: linear and angular velocity (1)

$$T_{sb}(t) = T(t) = \begin{bmatrix} R(t) & p(t) \\ 0 & 1 \end{bmatrix} \rightsquigarrow \dot{T} = \begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix}$$

$$T^{-1}\dot{T} = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} R^T\dot{R} & R^T\dot{p} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} [\omega_b] & v_b \\ 0 & 0 \end{bmatrix}$$

ang. vel in {b}

linear vel in {b}

ω_b and v_b gives a 6-parameter rep of a body twist: spatial vel in {b}

$$\mathcal{V}_b = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} \in \mathbb{R}^6$$

capital V

Moving Frames: linear and angular velocity (2)

$\underbrace{\text{bracket notation:}}_{\text{matrix rep of twists associated w/ } SE(3)}$ $[\mathcal{V}_b] = \underbrace{\begin{bmatrix} \overbrace{[\omega_b]}^{so(3)} & v_b \\ 0 & 0 \end{bmatrix}}_{se(3)} \in se(3)$

1. $[\overset{\in \mathbb{R}^6}{\mathcal{V}_b}]$ and $[\overset{\in \mathbb{R}^3}{\omega}]$ are different ops!

2. $[\mathcal{V}_b]$ is not skew symmetric

Linear and angular velocity in reference frame

$[\mathcal{V}_s] =$

$$\dot{T}T^{-1} = \begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \dot{R}R^T & \dot{p} - \dot{R}R^T p \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} [\omega_s] & v_s \\ 0 & 0 \end{bmatrix}$$

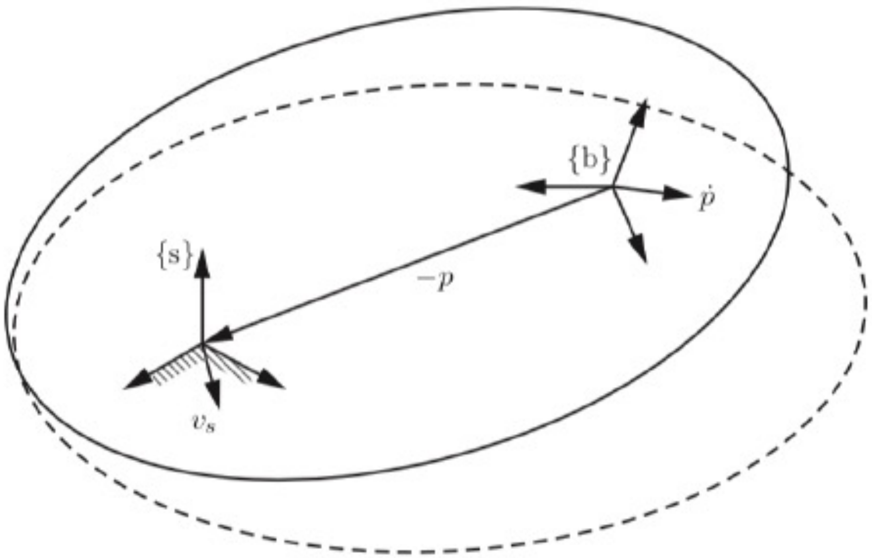
spatial twist : spatial vel in $\{s\}$ frame

$$\mathcal{V}_s = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix}$$

$$v_s = \dot{p} - \underbrace{\dot{R}R^T} p = \underbrace{\dot{p}}_{\text{linear componet}} \underbrace{- \omega_s \times p}_{\text{motion induced is } \{s\} \text{ by displacement}}$$

Physical interpretation of v_s

- Assume that a very large rigid body is attached to the frame {b}, and is large enough to contain the origin of {s}
- What is the velocity of the point of this body as the origin of {s}?
- There are two components:
  - $\dot{p}$: the motion of the body
  - $\omega_s \times (-p)$: the rotation of the body



Twists

$$[\mathcal{V}_b] = T^{-1}\dot{T} \quad \text{and} \quad [\mathcal{V}_s] = \dot{T}T^{-1}$$

$$[\mathcal{V}_b] = T^{-1}[\mathcal{V}_s]T \quad \text{and} \quad [\mathcal{V}_s] = T[\mathcal{V}_b]T^{-1}$$

$$[\mathcal{V}_b] = \begin{bmatrix} R[\omega_b]R^T & -R[\omega_b]R^T p + Rv_b \\ 0 & 0 \end{bmatrix} \overset{?}{\rightsquigarrow} \mathcal{V}_b \doteq [?]\mathcal{V}_s$$

trick: $[\omega]p = \omega \times p = -[p]\omega$ + some manip

$$\begin{bmatrix} \omega_s \\ v_s \end{bmatrix} = \underbrace{\begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix}}_{[Ad_T]} \begin{bmatrix} \omega_b \\ v_b \end{bmatrix}$$

$so(3)$

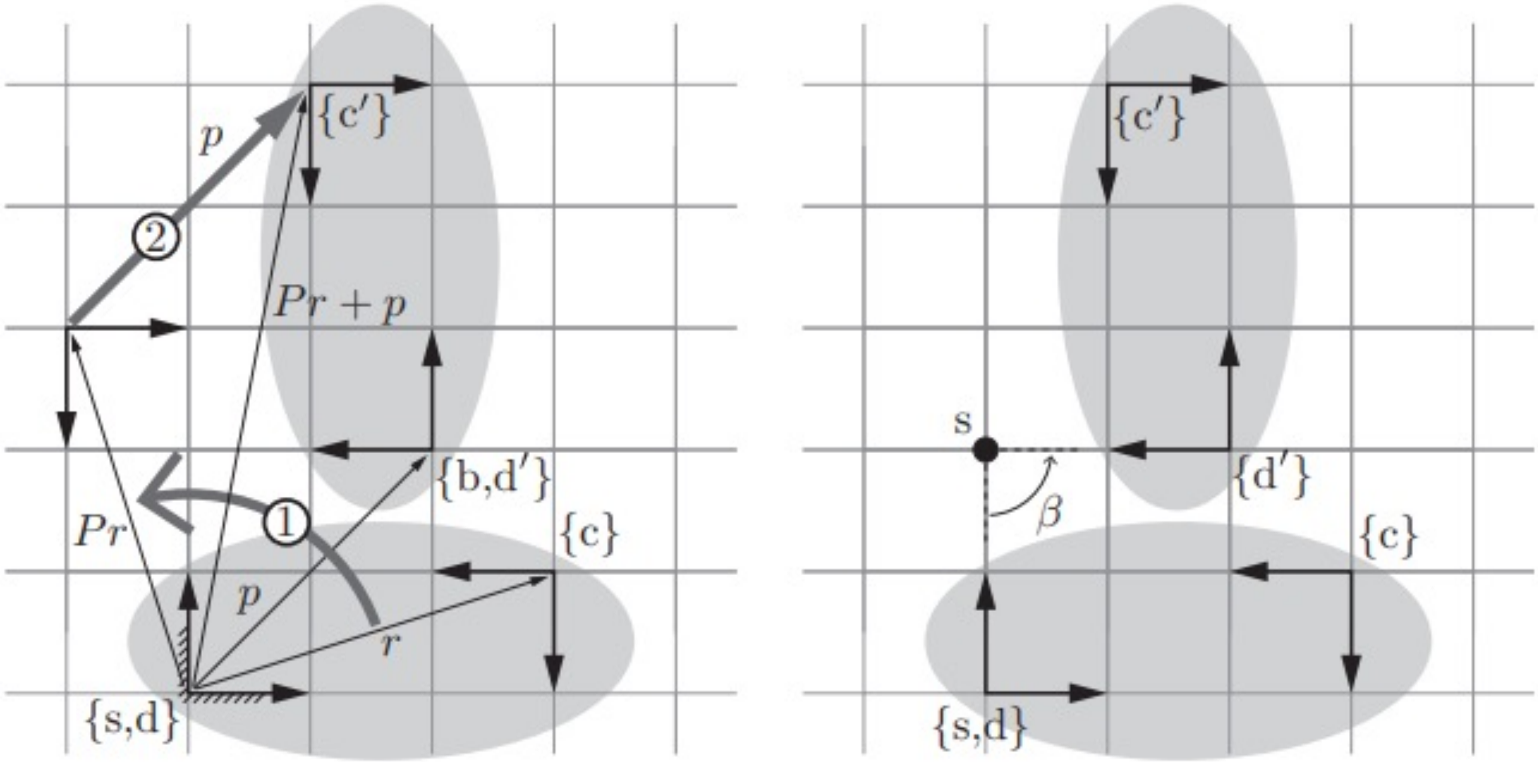
$[Ad_T] \rightarrow$ adjoint map rep

Adjoint Map associated with T: $Ad_T(\mathcal{V})$

$$\mathcal{V}' = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \mathcal{V} = [Ad_T]\mathcal{V}$$

- By construction, $\mathcal{V}' = Ad_T(\mathcal{V}) \Leftrightarrow [\mathcal{V}'] = T[\mathcal{V}]T^{-1}$
- Properties:
  1. $Ad_{T_1}\left(Ad_{T_2}(\mathcal{V})\right) = [Ad_{T_1}][Ad_{T_2}]\mathcal{V} = Ad_{T_1 T_2}(\mathcal{V})$
  2. $\left[Ad_{T_1^{-1}}\right] = [Ad_{T_1}]^{-1}$

Screw Motions



2D Screw Motions

- Any rigid motion in the plane can be represented by a rotation around a well-chosen center
- We can encode it with (β, s_x, s_y), where (s_x, s_y) is the position of the center of the rotation, and β the angle

2D Screw Motions

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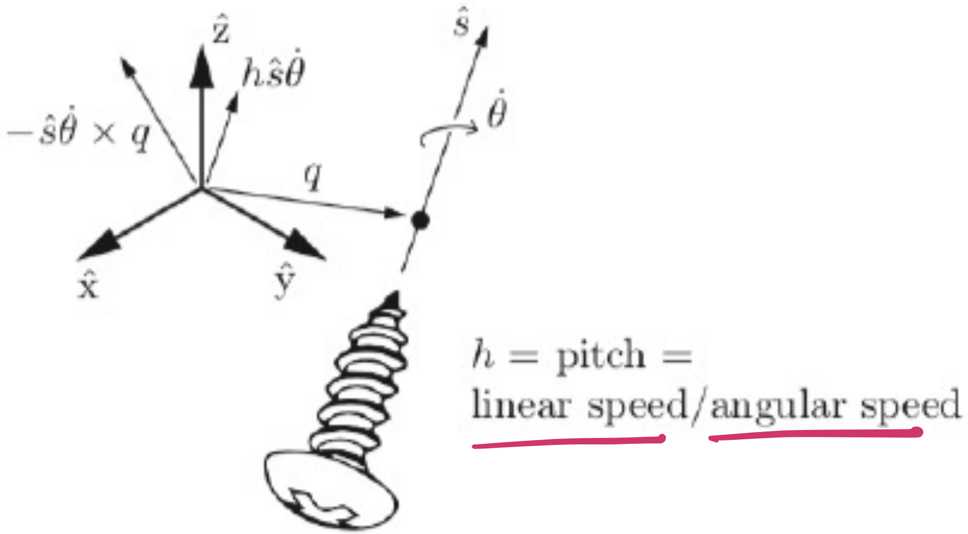
- Recall that the angular velocity ω can be viewed as $\hat{\omega}\dot{\theta}$, where $\hat{\omega}$ is the unit rotation axis and $\dot{\theta}$ is the rate of rotation
- A twist $\mathcal{V}$ can be interpreted in terms of a screw axis $\mathcal{S}$ and a velocity $\dot{\theta}$ about that axis

Screw motions in 3D

Chasles-Mozzi Theorem:
Any displacement in 3D can be represented by a rotation and translation about the same axis, referred to as a screw motion



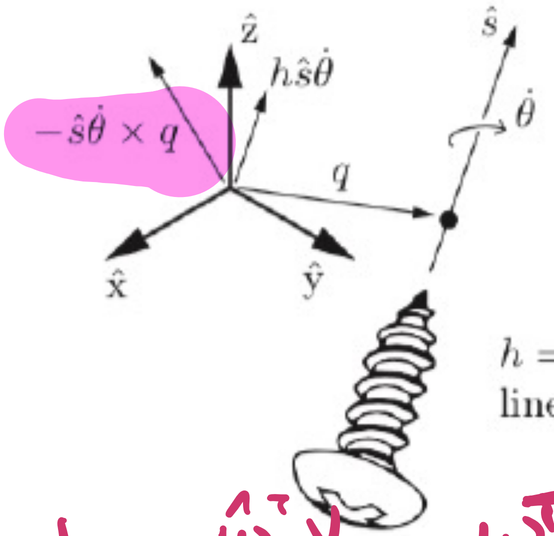
$q \in \mathbb{R}^3$ is any point along the axis

$\hat{s}$ is a unit vector in the direction of the axis

h is the **screw pitch**, which is the ratio between the linear and angular speed along axis

We write this as $\mathcal{S} = \{q, \hat{s}, h\}$

Screw motions and Twists



h = pitch = linear speed/angular speed

screw to twist

$$\mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix} = \begin{bmatrix} \hat{s}\dot{\theta} \\ h\hat{s}\dot{\theta} - \hat{s}\dot{\theta} \times q \end{bmatrix}$$

twist to screw:

$$\hat{s} = \omega / \|\omega\| = \hat{\omega}, \quad \dot{\theta} = \|\omega\|, \quad h = \frac{\hat{\omega}^T v}{\dot{\theta}} = \omega^T v / \|\omega\|^2$$

assume $\omega \neq 0$, find q s.t.

$$v = h\hat{s}\dot{\theta} - \hat{s}\dot{\theta} \times q \rightsquigarrow q = \frac{\hat{s} \times v}{\dot{\theta}}$$

Screw axis

Exponential Coordinates for Rigid Body Motions

What does the screw axis capture?

- A twist for 1 second around the screw axis at angular velocity θ, then the resulting frame is T
- A twist for θ seconds around the screw axis at angular velocity 1, then the resulting frame is T

Example of Screw Motion for Transformations

Find the pose that is produced by a given screw

Frame 0 is fixed in space and frame 1 is fixed in a body. The current pose of frame 1 in the coordinates of frame 0 is M. The body undergoes a body screw motion with axis $\mathcal{B}$ and magnitude θ. Find the new pose T_{01} of frame 1 in the coordinates of frame 0.

matlab python

```
M = [0.03262475 -0.39471442 0.91822445 -0.58853212; -0.21629944 -0.8997297
B = [0.00000000; 0.00000000; 0.00000000; -0.67130813; -0.02965809; 0.74058
theta = 1.29638080;
```

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matlab python

```
M = [0.03262475 -0.39
B = [0.00000000; 0.00
theta = 1.29638080;
```

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Note that my code snippet is in Julia!

[1]:
```
M = [0.03262475 -0.39471442 0.91822445 -0.58853212; -0.21629944 -0.89972977 -0.37907900
B = [0.00000000; 0.00000000; 0.00000000; -0.67130813; -0.02965809; 0.74058477];
theta = 1.29638080;

bb= [0 -B[3] B[2] B[4];
     B[3] 0 -B[1] B[5];
     -B[2] B[1] 0 B[6];
     0 0 0 0];
tt=exp(bb*theta);
T=M*tt
```

[1]:
```
4×4 Array{Float64,2}:
  0.0326248  -0.394714   0.918224   0.27982
 -0.216299   -0.89973   -0.379079   0.61705
  0.975782   -0.186244  -0.11473   -0.285795
  0.0         0.0        0.0        1.0
```

How to compute?

Proposition 3.25. *Let $\mathcal{S} = (\omega, v)$ be a screw axis. If $\|\omega\| = 1$ then, for any distance $\theta \in \mathbb{R}$ traveled along the axis,*

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} e^{[\omega]\theta} & \left(I\theta + (1-\cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2\right) v \\ 0 & 1 \end{bmatrix}. \tag{3.88}$$

If $\omega = 0$ and $\|v\| = 1$, then

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix}.$$

Compute $e^{[\omega]\theta}$ just like any matrix exponential!

$$e^{[\omega]\theta} = I + [\omega]\theta + [\omega]^2 \frac{\theta^2}{2!} + [\omega]^3 \frac{\theta^3}{3!} + \cdots$$

$$e^{[\omega]\theta} = I + \sin\theta\,[\omega] + (1-\cos\theta)[\omega]^2$$

Logarithm of rigid body motions

Given $T(R, p) \in SE(3)$, find $\mathcal{S} = [\omega \quad v]^{\mathsf{T}}$ and θ such that:

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \rightarrow [\mathcal{S}]\theta = \begin{bmatrix} [\omega]\theta & v\theta \\ 0 & 0 \end{bmatrix}$$

Define the logarithm of T

1. If $R = I$, then $\omega = 0$ and $v = \frac{p}{\|p\|}$ and $\theta = \|p\|$
2. Find ω, θ using matrix logarithm of R (see earlier slide!)
   Then, $v = G^{-1}(\dot{\theta})p$, where $G^{-1}(\dot{\theta}) = \frac{1}{\theta} I - \frac{1}{2}[\omega] + \left(\frac{1}{\theta} - \frac{1}{2}\cot\left(\frac{\theta}{2}\right)\right)[\omega]^2$

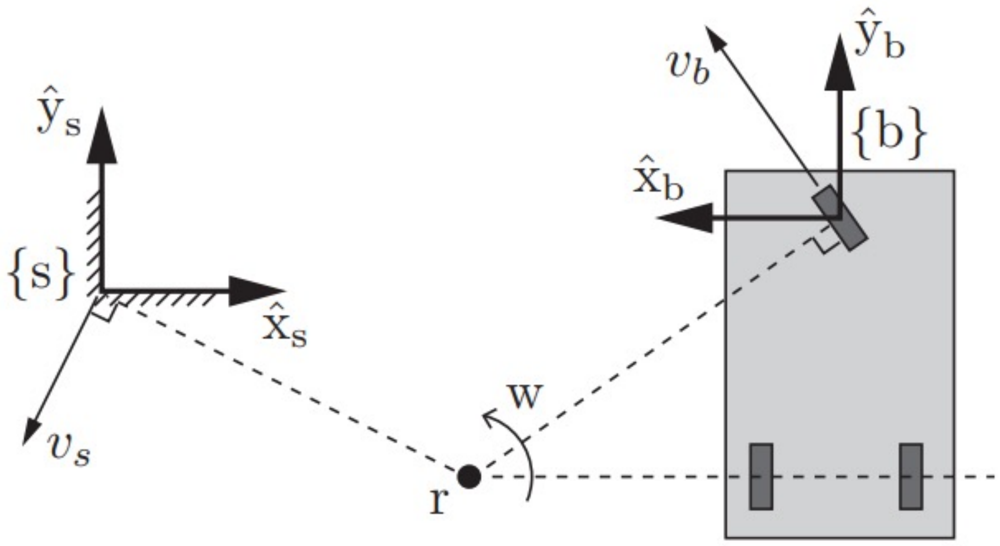
Now we can go back and forth between exponential coordinates and transformations!

Summary

- Used transformations to define spatial velocities as **body and spatial twists**
- Introduced **screw motions** and the screw interpretation of a twist
- The **screw axis $\mathcal{S}$** was defined and used to define the **exponential coordinates of homogeneous transformations**
- Connections between last lecture and this lecture:
  - Transformation matrix T is analogous to rotation matrix R
  - A screw axis $\mathcal{S}$ is analogous to rotation axis $\widehat{\omega}$
  - A twist $\mathcal{V}$ can be expressed as $\mathcal{S}\dot{\theta}$ and is analogous to angular velocity $\omega = -\widehat{\omega}\dot{\theta}$
  - Exponential coordinates $\mathcal{S}\theta \in \mathbb{R}^6$ for rigid body motions are analogous to exponential coordinates $\widehat{\omega}\theta \in \mathbb{R}^3$

Extra Slides

Twist Visualization



The twist corresponding to the instantaneous motion of the chassis of a three-wheeled vehicle can be visualized as an angular velocity w about the point r.