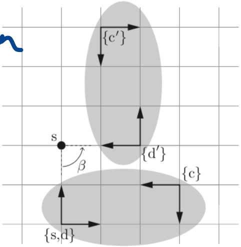
Screw motion

All* motions in 2D are rotations (about a well-chosen center)

* rigid, orientation preserving.

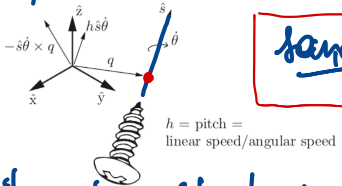


- 2D Screw motion: any rigid motion in the plane can be represented by a rotation around a well-chosen center.
- We can encode it with (β, s_x, s_y), where (s_x, s_y) is the position of the center of the rotation, and β the angle. Here, $\beta = \pi/4$ and $s = (0, 2)$

Screw motion

in 3D: rotation & translation about the same axis.

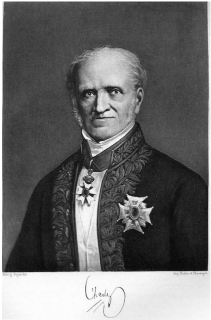
note: by convention, the translation amount is given by $\theta . h$ (eg, if we translate by 10 cm & rotate by 10°, then $h = 1$ cm/deg, $\theta = 10°$



- **Chasles-Mozzi Theorem**: Any displacement in 3D can be represented by a rotation + translation about same axis. This is called a screw motion.
- Data for a screw: Axis of rotation: $q \in \mathbb{R}^3 + \hat{s} \in S^2$; pitch h : ratio linear/angular speed; $\dot{\theta}$: angular velocity.
- Rotation of θ rad results in translation of $h\theta$ along axis.
- Very useful to represent motion of revolute and prismatic joints.
- Write as $\mathcal{S} = \{q, \hat{s}, h\}$. 6 parameters; well suited to describe joints

Ball and Chasles

Chasles

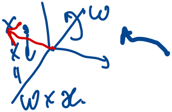


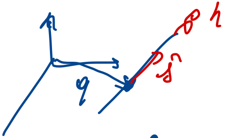
- Rodrigues and Chasles took the entrance exam to Polytechnique/Normale at the same time, finishing first and second respectively. Rodrigues did not use it and elected to go to La Sorbonne.

Screw motion → twist

$\{q, \hat{s}, h\}$

recall: $\dot{x} = \omega \times x$

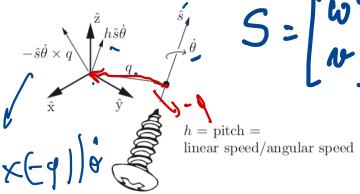




$S = \begin{bmatrix} \omega \\ v \end{bmatrix}$

$\omega_s = \hat{s}$

$v_s = (-\hat{s} \times q)\dot{\theta} + h\hat{s}\dot{\theta}$



$(\hat{s} \times (-q))\dot{\theta}$

- How to represent a screw motion as a twist? Take $\mathcal{S} = \{q, \hat{s}, h\}$.
  - Rotational velocity: $\omega = \hat{s}\dot{\theta}$.
  - Linear velocity : velocity at origin of frame= translation along the screw axis + linear motion at origin induced by rotation of screw axis:

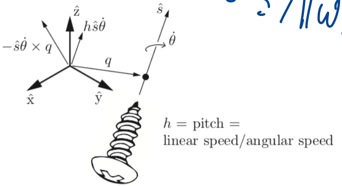
$$v = h\hat{s}\dot{\theta} - \hat{s}\dot{\theta} \times q.$$

Twist → Screw motion

$\begin{pmatrix}\omega_s \\ v_s\end{pmatrix} \overset{?}{\to} (q, \hat{s}, h)$

$V_s = -(\hat{s} \times q)\dot{\theta} + h\hat{s}\dot{\theta}$

$\omega_s / \|\omega_s\| = \hat{s}$



- How to represent a twist as a screw motion? Take $\mathcal{V} = [\omega, v]^\top$
  - Rotation+pitch: $\hat{s} = \omega/\|\omega\| = \bar{\omega}$, $\dot{\theta} = \|\omega\|$ and $h = \bar{\omega}^\top v/\dot{\theta} = \omega^\top v/\|\omega\|^2$.
  - Offset q: Assuming $\omega \neq 0$ (not pure translation, so that there is a meaningful rotation axis), need to find q so that $v = h\hat{s}\dot{\theta} - \hat{s}\dot{\theta} \times q$. Cross product on left by $\hat{s}$ yields

$$q = \frac{\hat{s} \times v}{\dot{\theta}}.$$

$$v_s = -(\hat{s} \times q)\dot{\theta} + h\,\hat{s}\,\dot{\theta}$$

recall $\hat{s}^T v_s = \langle \hat{s}, v_s \rangle$

$= \hat{s} \cdot v_s$ } inner or dot product

<u>but</u>: $\hat{s} \times q$ is by definition

$\perp$ to both q & $\hat{s}$!!

$$\Rightarrow \underset{(\,\cdots\,)\,(\vdots)}{\hat{s}^T v_s} = h \underbrace{\hat{s}^T \hat{s}}_{\|\hat{s}\|^2 = 1} \dot{\theta} = h\dot{\theta}$$

$$\rightarrow \quad h = \frac{\hat{s}^T v_s}{\dot{\theta}} = \frac{[\omega_s / \|\omega_s\|]^T v_s}{\|\omega_s\|}$$

$$= \frac{\omega_s^T v_s}{\|\omega_s\|^2}$$

$$\begin{pmatrix} \omega_s \\ v_s \end{pmatrix} \rightsquigarrow \begin{aligned} \hat{s} &= \omega_s / \|\omega_s\| \\ \dot{\theta} &= \|\omega_s\| \\ h &= \frac{\omega_s^T v_s}{\|\omega_s\|^2} \end{aligned}$$

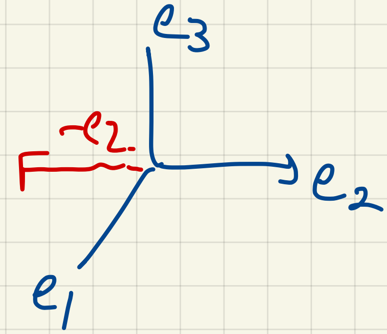
How about q?

$$v_s = -(\hat{s} \times q)\dot{\theta} + h \hat{s} \dot{\theta}$$

$$\hat{s} \times v_s = \hat{s} \times (-\hat{s} \times q)\dot{\theta} + \underbrace{\hat{s} \times h \hat{s} \dot{\theta}}_{0}$$

example:

$$\underbrace{e_1 \times \underbrace{(e_1 \times e_2)}_{e_3}}_{-e_2}$$



$$= -(\hat{s} \times (\hat{s} \times q))\dot{\theta} + 0$$

$$= -(-q)\dot{\theta}$$

$$= q\dot{\theta}$$

$$\Rightarrow q = (\hat{s} \times v_s)/\|\omega_s\|$$

$\therefore$ To formally show that

$\hat{s} \times (\hat{s} \times q) = -q$ for any normalized $\hat{s}$ non // to q, we determinant formula.

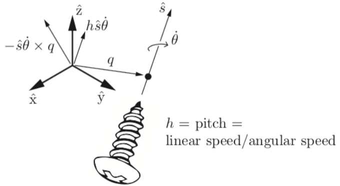
$$\begin{pmatrix} \omega_s \\ v_s \end{pmatrix} \rightarrow \quad \hat{s} = \omega_s / \|\omega_s\|$$

$$\dot{\theta} = \|\omega_s\|$$

$$h = \frac{\omega_s^T v_s}{\|\omega_s\|^2}$$

$$q = \frac{\omega_s \times v_s}{\|\omega_s\|^2}$$

Screw motion



- For a given reference frame, we denote by $\mathcal{S}$ the *screw axis* of the screw motion $\{q, \hat{s}, \dot{\theta}\}$

$$\mathcal{S} = \begin{bmatrix} \omega \\ v \end{bmatrix}$$

$S = \begin{pmatrix} 0 \\ v \end{pmatrix}$ ∴

with either 1: $\|\omega\| = 1$ or 2: $\omega = 0$ and $\|v\| = 1$.
Case 1: then $v = -\omega \times q + h\omega$
Case 2: corresponds to a translation along v: (h is infinite) $\rightarrow \omega = 0$.

Exponential coordinates for Rigid-Body motions

- Recall: for a rotation $R \in SO(3)$, its *exponential coordinates* $(\bar{\omega}, \theta)$ are so that $R = e^{[\bar{\omega}]\theta}$. We want to do the same for rigid motions (i.e., rotation + translation)
- The matrix $[\bar{\omega}]$ was skew-symmetric: $[\bar{\omega}] \in \mathfrak{so}(3)$.
- We defined

$$\exp : \mathfrak{so}(3) \to SO(3) : A \to I + A + A^2/2! + \cdots$$
$$\log : SO(3) \to \mathfrak{so}(3) : R \to \frac{1}{2\sin\theta}(R - R^\top)$$

Exponential coordinates for Rigid-Body motions

- By analogy, we define the exponential coordinates for a homogeneous transformation T as $\mathcal{S}\theta \in \mathbb{R}^6$, where $\mathcal{S}$ is the screw axis and θ the distance around the screw axis to take a frame from I to T.
- Recall that if $\mathcal{S} = (\omega, v)$ is with $\|\omega\| = 1$, then θ is an angle of rotation about the screw axis. If $\omega = 0$ and $\|v\| = 1$, then θ is the distance travelled along the screw axis.
- Denote by $SE(3)$ the space of homogeneous transformations T, and by $\mathfrak{se}(3)$ the space of their exponential coordinates. We want to define

$$\begin{aligned} \exp : &\mathfrak{se}(3) \to SE(3) : A \to I + A + A^2/2! + \cdots \\ \log : &SE(3) \to \mathfrak{se}(3) : T \to [\mathcal{S}]\theta \in \mathfrak{se}(3) \end{aligned}$$

Exponential coordinates for Rigid-Body motions

For $\mathcal{S} = [\omega, v]^\top$, we introduce the 4×4 matrix:

$$[\mathcal{S}] := \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix}$$

- As before, we can find a simple expression for the exponential. We have (recall: $\omega = \overline{\omega}\theta$)

$$\begin{aligned} e^{[\mathcal{S}]\theta} &= I + [\mathcal{S}]\theta + [\mathcal{S}]^2\theta^2/2! + [\mathcal{S}]^3\theta^3/3! + \cdots \\ &= \begin{bmatrix} e^{[\overline{\omega}]\theta} & G(\theta)v \\ 0 & 1 \end{bmatrix} \end{aligned}$$ $\rightsquigarrow$ confused part.

  where $G(\theta) = I\theta + [\overline{\omega}]\theta^2/2! + [\overline{\omega}]^2\theta^3/3! + \cdots$.
- Recall that $[\overline{\omega}]^3 = -[\overline{\omega}]$. We obtain

$$\begin{aligned} G(\theta) &= I\theta + (\theta^2/2! - \theta^4/4! + \cdots)[\overline{\omega}] + (\theta^3/3! - \theta^5/5! + + \cdots)[\overline{\omega}]^2 \\ &= I\theta + (1 - \cos\theta)[\overline{\omega}] + (\theta - \sin\theta)[\overline{\omega}]^2 \end{aligned}$$

Logarithm of Rigid-Body motions

- Given $T = (R, p) \in SE(3)$, we need to find $\mathcal{S} = (\overline{\omega}, v)$ and θ so that

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix},$$

  which implies that

$$[\mathcal{S}]\theta = \begin{bmatrix} [\overline{\omega}]\theta & v\theta \\ 0 & 0 \end{bmatrix}$$

  is a logarithm of T.
- Case 1: If $R = I$ (pure translation), then set $\overline{\omega} = 0$, $v = p/\|p\|$ and $\theta = \|p\|$.
- Case 2: **1.** Obtain $\overline{\omega}, \theta$ using the matrix logarithm of R. **2.** Set

$$v = G^{-1}(\theta)p$$

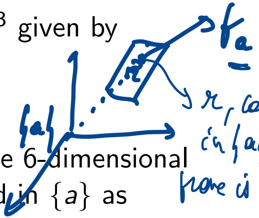
  where

$$G^{-1}(\theta) = \frac{1}{\theta}I - \frac{1}{2}[\overline{\omega}] + \left(\frac{1}{\theta} - \frac{1}{2}\cot\frac{\theta}{2}\right)[\overline{\omega}]^2.$$

Wrenches

- Let $\{a\}$ be a ref. frame and r_a a point in a rigid body. Let the vector $f_a \in \mathbb{R}^3$ represent a force acting on the body. Assume it is a point force acting at r_a.
- The force creates a *torque or moment* $m_a \in \mathbb{R}^3$ given by

$$\boxed{m_a := r_a \times f_a.}$$

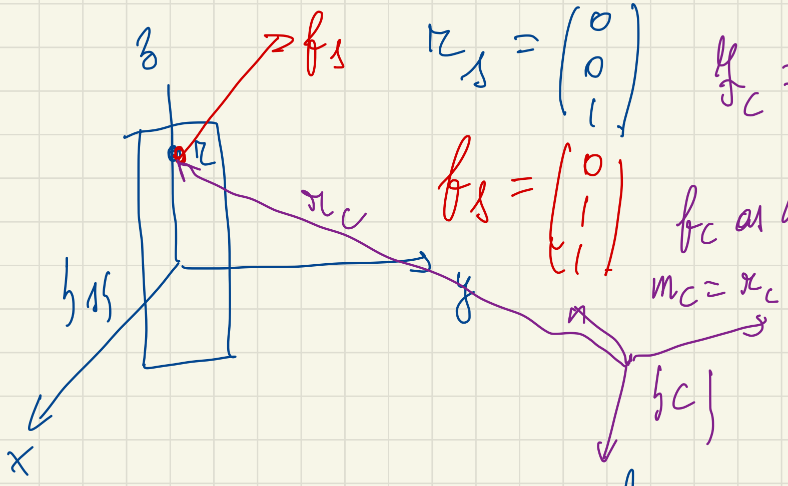


r_a coord in $\{a\}$ frame is x_a

- We merge the force and the moment in a single 6-dimensional vector, called *spatial force* or *wrench*, expressed in $\{a\}$ as

$$\mathcal{F}_a := \begin{bmatrix} m_a \\ f_a \end{bmatrix}. \quad \in \mathbb{R}.$$

- A wrench can contain only a torque and no force: it is called a *pure moment*.
- If there are several forces/moments acting on the body, we can add their corresponding wrenches.



$$r_s = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad f_s = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \qquad \mathcal{F}_c = \begin{pmatrix} m_c \\ f_c \end{pmatrix}$$

f_c as usual

$m_c = r_c \times f_c$

$$\mathcal{F}_s = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \qquad m_s = r_s \times f_s = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

How to obtain $\mathcal{F}$ in a different frame?

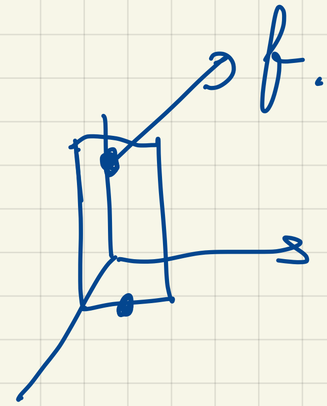
recall: $\mathcal{V}_a = Ad_{T_{ab}} \mathcal{V}_b$ — pos. of orig. of b in a frame

$$\mathcal{V}_a = \begin{pmatrix} \omega_a \\ v_a \end{pmatrix} \qquad \mathcal{V}_b = \begin{pmatrix} \omega_b \\ v_b \end{pmatrix} \qquad T_{ab} = \begin{pmatrix} R_{ab} & P_a \\ 0\,0\,0 & 1 \end{pmatrix}$$

$$Ad_T = \begin{pmatrix} R & 0 \\ [p]R & R \end{pmatrix} \qquad Ad_{T_{ab}}[\mathcal{V}_b] = T_{ab}[\mathcal{V}_b]T_{ba}$$

Power is independent of frame.

"force · speed"



"$\mathcal{F}_s^T \mathcal{V}_s$ = power."

$$= \mathcal{F}_c^T \mathcal{V}_c$$

$$\mathcal{F}_s^T \mathcal{V}_s = \mathcal{F}_c^T Ad_{T_{cs}} \mathcal{V}_s$$

or $\mathcal{F}_s^T Ad_{T_{sc}} \mathcal{V}_c = \mathcal{F}_c^T \mathcal{V}_c$.

↳ true for all $\mathcal{V}_c$! $\Rightarrow$

$$\mathcal{F}_s^T Ad_{T_{sc}} = \mathcal{F}_c^T$$ "duality"

$$\Rightarrow \boxed{\mathcal{F}_c = \left[Ad_{T_{sc}}\right]^T \mathcal{F}_s}$$

$$\left[Ad_{T_{sc}}\right]^T \neq Ad_{T_{cs}} \neq Ad_{T_{sc}^T}$$

Wrenches

- Given 2 frames, a and b, and want to express the wrench in both frames: $\mathcal{F}_a$, $\mathcal{F}_b$.
- Denote by $\mathcal{V}_a$, $\mathcal{V}_b$ the motions induced by the wrench in frames a and b: we know how to relate them: $\mathcal{V}_b = \mathrm{Ad}_{T_{ba}} \mathcal{V}_a$
- Recall: dot product of velocity and force is a power!
- Power is independent from choice of frame!

$$\begin{aligned} \mathcal{V}_b^\top \mathcal{F}_b &= \mathcal{V}_a^\top \mathcal{F}_a \\ &= (\mathrm{Ad}_{T_{ab}} \mathcal{V}_b)^\top \mathcal{F}_a \\ &= \mathcal{V}_b^\top (\mathrm{Ad}_{T_{ab}})^\top \mathcal{F}_a \end{aligned}$$

- Since the previous equation holds for *all* $\mathcal{V}_b$, we have

$$\mathcal{F}_b = (\mathrm{Ad}_{T_{ab}})^\top \mathcal{F}_a$$

$$Ad_{T_{sc}} = \begin{bmatrix} R_{sc} & 0 \\ [p_s] R_{sc} & R_{sc} \end{bmatrix}$$ where $T_{sc} = \begin{bmatrix} R_{sc} & p_s \\ 000 & 1 \end{bmatrix}$

$$\left(Ad_{T_{sc}}\right)^T = \begin{bmatrix} R_{cs} & -R_{cs}[p_s] \\ 0 & R_{cs} \end{bmatrix}$$

$R_{cs} \underbrace{[p_s]^T}_{-[p_s]}$ "

because $R_{sc}^T = R_{cs} = R_{sc}^{-1}$

$[p_s]^T = -[p_s]$

because if $p_s = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$ then $[p_s] = \begin{bmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{bmatrix}$

note: recall f_c is f_s in {c} frame

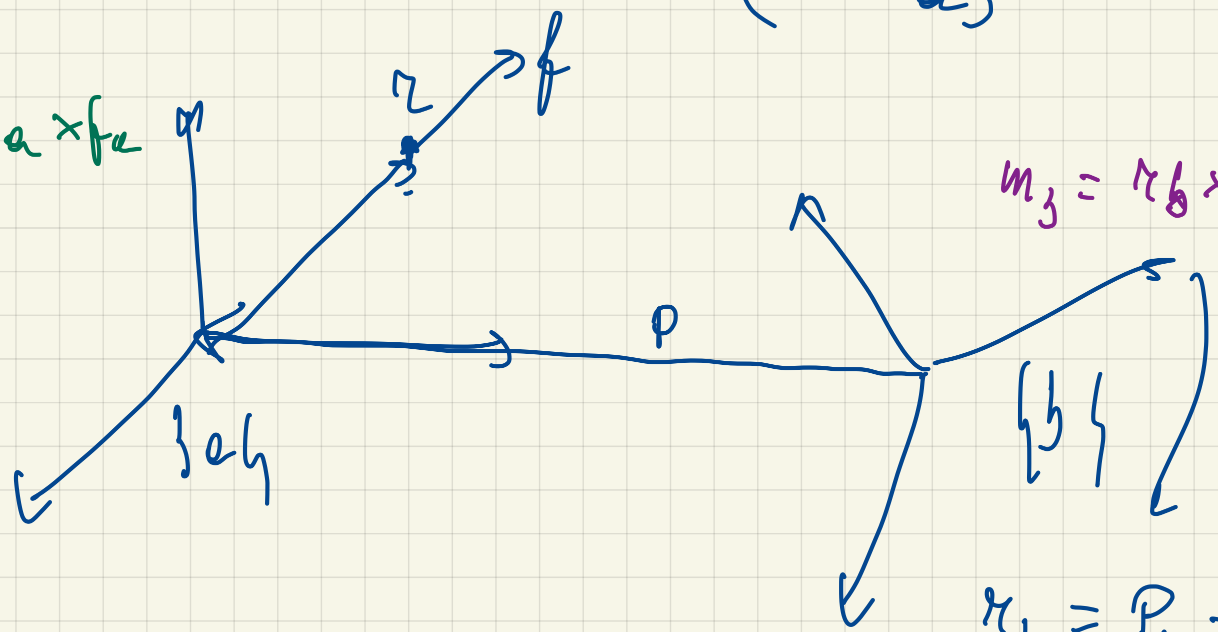
thus $f_c = R_{cs} f_s$.

Introduction to Robotics
Forward Kinematics

MA Belabbas

Exercise: verify $(Ad_{T_{ab}})^T \mathcal{F}_a = \mathcal{F}_b$

$m_a = r_a \times f_a$



$m_b = r_b \times f_b$

$r_b = p_b + R_{ba} r_a$

"$[p] v = p \times v$"

$[S_1]$ $[S_2]$ $[S_3]$

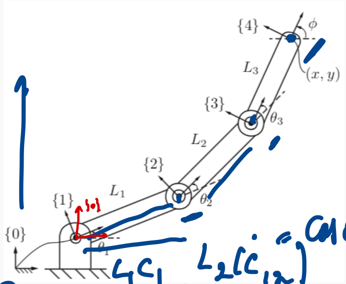
$$\dot{T}_{sb} = [\mathcal{V}_s] T_{sb}$$

$$\rightsquigarrow T_{sb} = \underbrace{\boxed{e^{[\mathcal{V}_s]\theta}}}_{T_{01}} \underbrace{T_{sb}(0)}_{T_1}.$$

Forward kinematics of open chains

$2D : (x, y, \phi)$

$\phi = \theta_1 + \theta_2 + \theta_3$



$\theta_3 =$

$L_1 c_1 \quad L_2 (c_{12}) = \cos \theta_1 + \theta_2$

$\theta_1 \; \theta_2 \; \theta_3 \rightsquigarrow$ pos. end effector

- The *forward kinematics* of a robot refers to the calculation of the position and orientation of its effector frame from its joint coordinates.
- Here, simple trigonometry yields

$$\rightsquigarrow \left| \begin{aligned} x &= L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) + L_3 \cos(\theta_1 + \theta_2 + \theta_3) . \\ y &= L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2) + L_3 \sin(\theta_1 + \theta_2 + \theta_3) . \end{aligned} \right.$$

Forward kinematics

- For more complex 3D mechanisms, a direct analysis as done in previous slide is too onerous.
- We describe two systematic, principle ways to perform forward kinematics: Product of Exponentials (PoE) and *Denavit-Hartenberg* (next lecture).
- What we can do using previous lectures: attaching frames 1,2,3,4 to the three links and end-effector respectively, and denoting by 0 the reference frame, we need T_{04}, which we can obtain as

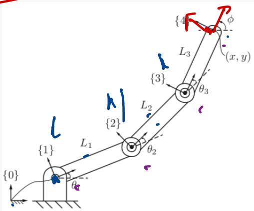
$$\boxed{T_{04} = T_{01}\,T_{12}\,T_{23}\,T_{34},} = \begin{bmatrix} \mathrm{Rot}(\theta_1+\theta_2+\theta_3, \hat{z}) & p \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix}$$

where $p = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$

and $T_{i(i+1)}$ are easy to derive.

$\{0\}$ = space frame $\{1\}$: frame attached to link 1

Forward kinematics: using homogeneous transformations



$$T_{01} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, T_{12} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & L_1 \\ \sin\theta_2 & \cos\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{23} = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 & L_2 \\ \sin\theta_3 & \cos\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, T_{34} = \begin{bmatrix} 1 & 0 & 0 & L_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$