Homogeneous Transformations: SE(3)

The **special Euclidean group** $\boldsymbol{SE}(\mathbf{3})$ is the set of $4{\times}4$ matrices of the form:

$$T = T(R, p) = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$$

Homogeneous Transformations: SE(3)

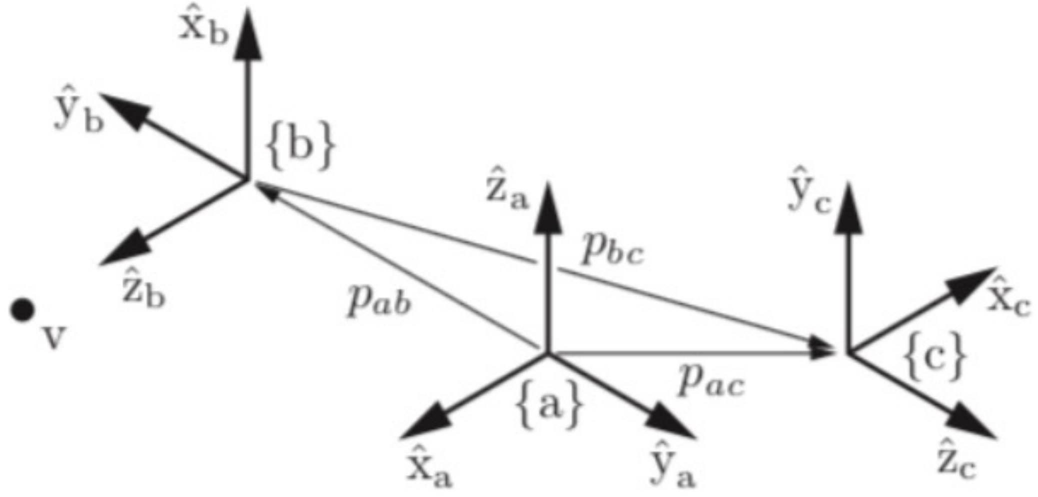
The **special Euclidean group $\boldsymbol{SE(3)}$** is the set of 4×4 matrices of the form:

$$T = T(R, p) = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$$

- The inverse of T is $T^{-1} = \begin{bmatrix} R^\top & -R^\top p \\ 0 & 1 \end{bmatrix} \in SE(3)$
- If T_1 and $T_2 \in SE(3)$, then $T_1 T_2 \in SE(3)$
- T satisfies: $\|Tx - Ty\| = \|x - y\|$ and

$$(Tx - Tz)^\top (Ty - Tz) = (x - z)^\top (y - z)$$

Representing a configuration with SE(3)



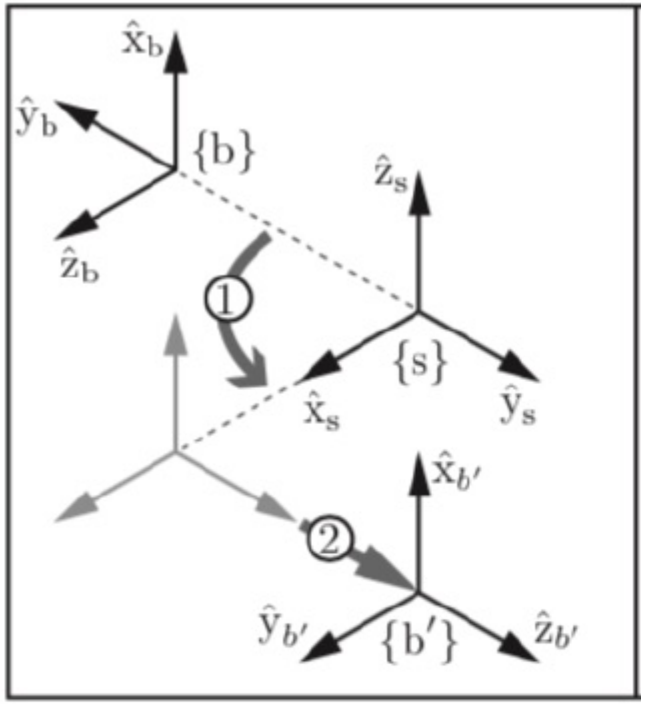
Each frame can represent a body frame in a multi-link mechanism

As before:

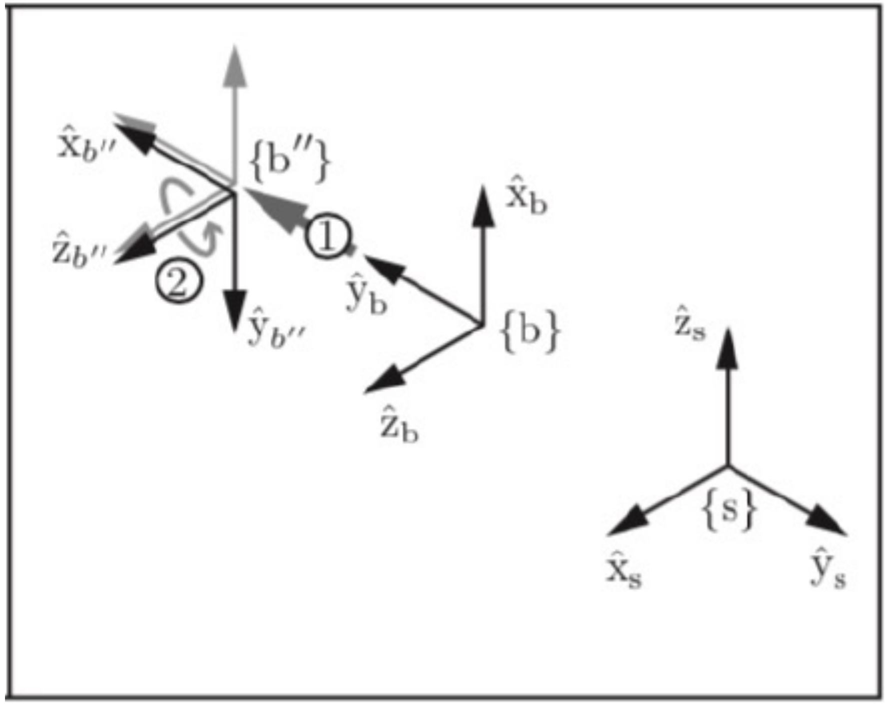
$$T_{ab}T_{bc} = T_{ac}$$

$$T_{ab}v_b = v_a$$

Example Displacement (1)

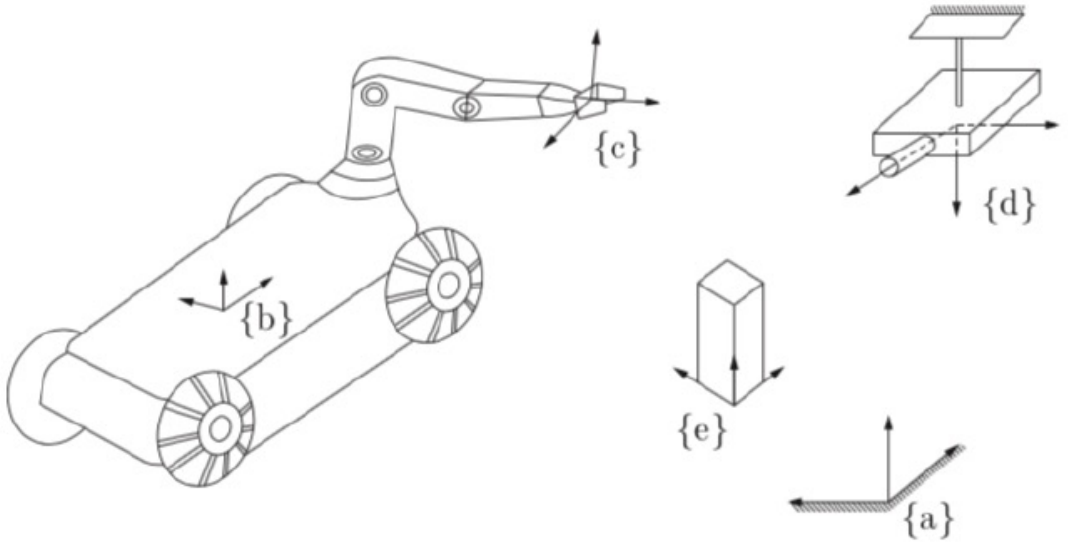


Example Displacement (2)



Mobile arm example

- Robot arm mounted on wheeled platform. Camera fixed to ceiling.
- {b} is body frame, {c} end-effector frame, {e} frame of object, and {a} is fixed frame.
- We assume the camera position and orientation in {a} is given.
- From camera measurements, you can evaluate the position and orientation of the body and the object in the camera frame.
- Since we designed our robot and have joint-angle estimates, we can obtain the end-effector position and orientation in the body frame.
- To pick up the object, we need the object position and orientation in the frame of our end-effector.



Summary

- Introduced **exponential coordinates** that allow us to parameterize rotations by the rotation axis $\hat{\omega}$ and the angle of rotation θ
- Walked through the **matrix logarithm** method, which tells us how to recover the exponential coordinates from a rotation matrix
- Started discussing **homogeneous transformations** where we not only look at rotations, but also translations in a single operation

Who is Michael Chasles?

- Rodrigues and Chasles took the entrance exam to Polytechnique/Normale at the same time, finishing first and second respectively
  - Rodrigues did not use it and elected to go to La Sorbonne
- The proof that a spatial displacement can be decomposed into a rotation and slide around and along a line is attributed to the astronomer and mathematician Giulio Mozzi (1763),
  - In Italy, the screw axis is traditionally called *asse di Mozzi*
- However, most textbooks refer to a subsequent similar work by Michel Chasles dating 1830
- Several other scholars/contemporaries of M. Chasles obtained the same or similar results around that time, including G. Giorgini, Cauchy, Poinsot, Poisson, and Rodrigues

Mobile arm example

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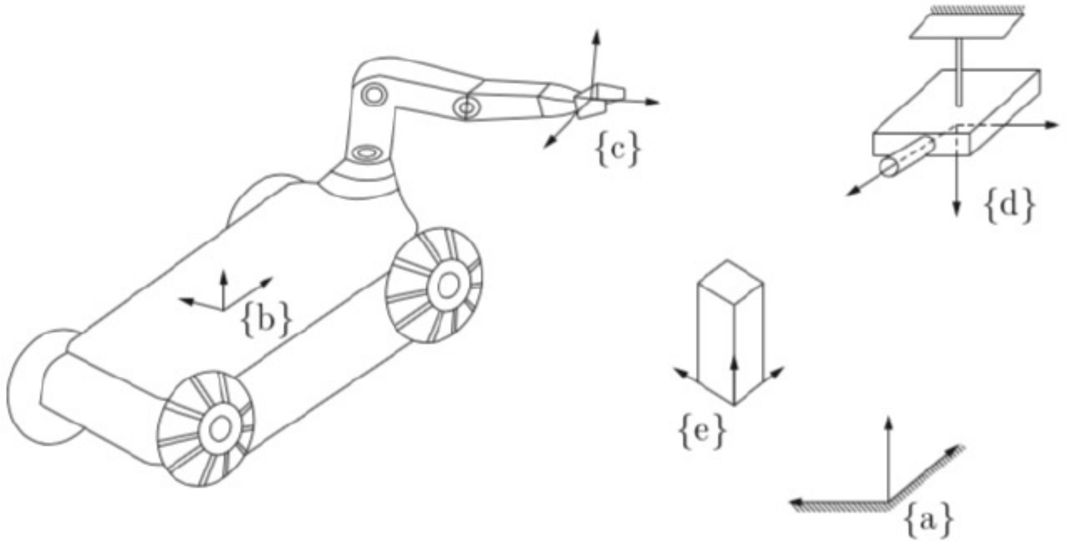
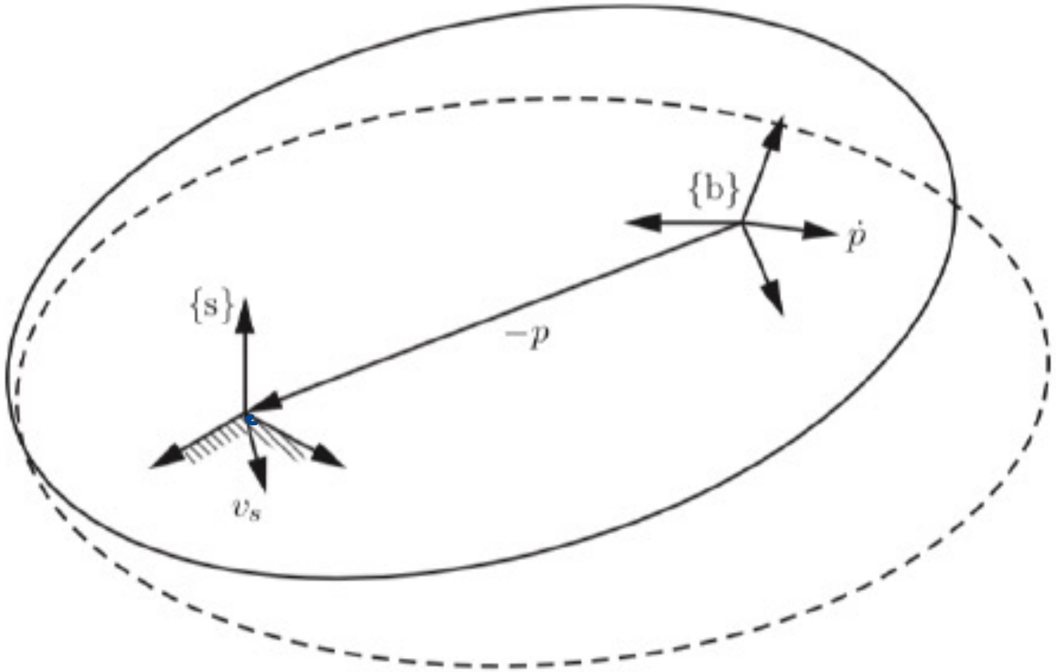


Physical interpretation of v_s

- Assume that a very large rigid body is attached to the frame {b}, and is large enough to contain the origin of {s}
- What is the velocity of the point of this body as the origin of {s}?
- There are two components:
  - $\dot{p}$: the motion of the body
  - $\omega_s \times (-p)$: the rotation of the body



* · Adjoint map. ∴ 9/23
· Screw theory. ∴
↳ · Forward Kinematics map

· $T_{sb} = \begin{bmatrix} R_{sb} & p_s \\ 0\,0\,0 & 1 \end{bmatrix}$

$$\dot{T}_{sb} = [\mathcal{V}_s]\, T_{sb}$$

$[\mathcal{V}_s] = \dot{T}_{sb}\, T_{bs} = \dot{T}_{ss}$

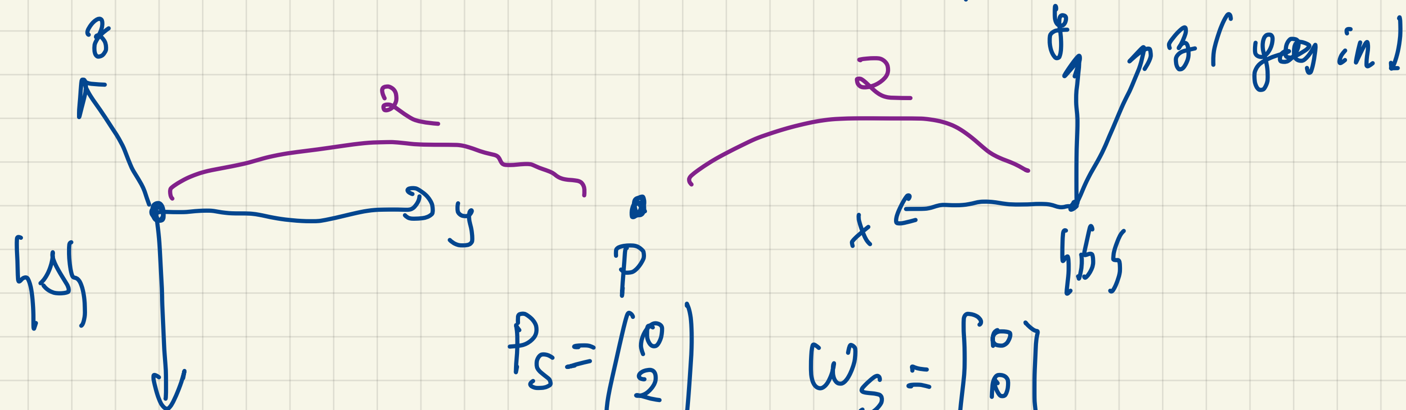
$$\dot{T}_{sb} = T_{sb}\, [\mathcal{V}_b]$$

$$\mathcal{V}_s = \begin{bmatrix} \omega_s \\ \\ v_s \end{bmatrix} \qquad \mathcal{V}_b = \begin{bmatrix} \omega_b \\ \\ v_b \end{bmatrix}$$

$$[\mathcal{V}_s] = \begin{bmatrix} [\omega_s] & v_1 \\ 0\,0\,0 & 0 \end{bmatrix} \qquad [\mathcal{V}_b] = \begin{bmatrix} [\omega_b] & v_b \\ 0\,0\,0 & 0 \end{bmatrix}$$

$$\omega_s = R_{sb}\,\omega_b$$

$$v_1 = \dot{p}_s - \omega_s \times p_s$$



$$p_s = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \qquad \omega_s = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathcal{V}_s = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} \qquad \mathcal{V}_b = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

v_b

$$v_1 = -\omega_s \times p = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

Adjoint Map associated with T

$$\dot{T}_{sb} = [\mathcal{V}_s] T_{sb} \qquad \dot{T}_{sb} = T_{sb} [\mathcal{V}_b]$$

$$\Longrightarrow \quad T_{sb}^{-1} [\mathcal{V}_s] T_{sb} = [\mathcal{V}_b]$$

$$Ad_{T_{bs}} [\mathcal{V}_s] = [\mathcal{V}_b]$$

by def. $Ad_X Y = X Y X^{-1}$ for X an invertible matrix, Y a matrix.

Ad is the adjoint map.

note: $\exists$ ad (also called adjoint)
do not confuse it with Ad.

$$ad_{[V_b]}[V_s] = [V_b][V_s] - [V_s][V_b]$$

↳ not used for now, appear in dynamics

∴ Exercise: $T_{sb} = \begin{bmatrix} R_{sb} & p_s \\ 000 & 1 \end{bmatrix}$

$$[V_b] = Ad_{T_{sb}}[V_s].$$

$$= T_{sb}^{-1}[V_s]T_{sb}$$

$$V_b = Ad_{T_{bs}} V_s \rightsquigarrow [V_b] = [Ad_{T_{bs}} V_s]$$

$$Ad_{T_{bs}} \to \begin{array}{cc} \begin{bmatrix} R_{bs} & 0 \\ [p]R & R \end{bmatrix} \\ \begin{array}{cc} X \quad & \quad Y \end{array} \end{array}$$

$$V_s = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} \qquad V_b = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix}$$

$$\omega_b = R_{bs}\,\omega_s$$

$$v_b = X\omega_s + Y v_s.$$

$$[V_s] = Ad_{T_{sb}} [V_b]$$

↳ conjugation

$$V_s = Ad_{T_{sb}} V_b$$

$T_{sb} = \begin{bmatrix} R_{sb} & p_s \\ 000 & 1 \end{bmatrix}$ ↳ $\begin{pmatrix} R_{sb} & 0 \\ [p_s] R_{sb} & R_{sb} \end{pmatrix}$ 6.

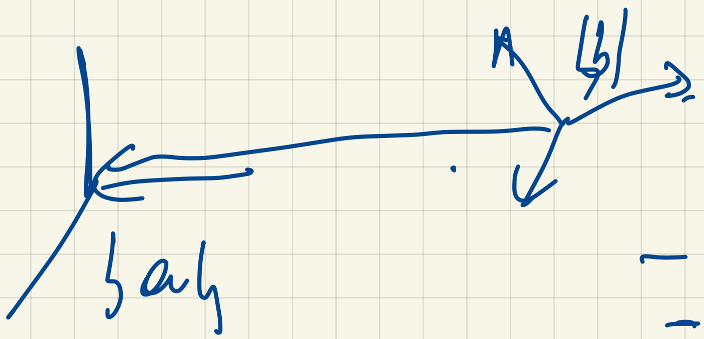
∴ Adjoint map : change of basis for twists.

$$[p] = \begin{bmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{bmatrix}$$

<u>example:</u> $[p_s] R_{sb} \, w_b = [p_s] w_s = p_s \times w_s$

$= -w_s \times p$

recall $T_{ab} = \begin{bmatrix} R_{ab} & P_a \\ 000 & 1 \end{bmatrix}$



T_{ba}

$= R_{ba} P_a$ '' $= P_b$

$T_{ab} = \Big[$ coord of {b}-axis in a frame   position of center of {b} frame in coord of {a} frame

---

$[\hat{\omega}] \quad \theta \quad \to \quad e^{[\hat{\omega}]\theta}$

$= I + \sin\theta\,[\hat{\omega}] + (1-\cos\theta)\,[\hat{\omega}]^2$

Rodrigues formula.

Exponential Coordinates for Rigid Body Motions

$\mathcal{V}_S = \begin{pmatrix} \omega_S \\ v_s \end{pmatrix}$

- Recall for rotation matrices: $R = e^{[\hat{\omega}]\theta}$
- What are exponential coordinates for T?

$\mathcal{S}\theta \in \mathbb{R}^6$, where $\mathcal{S}$ is the screw axis and θ is the distance about $\mathcal{S}$ to take frame from I to T

$$e^{[\mathcal{V}_S]\theta}$$

$$T = \underline{e^{[\mathcal{S}]\theta}} = I + [\mathcal{S}]\theta + [\mathcal{S}]^2 \frac{\theta^2}{2!} + \cdots$$

solution of

$\dot{T}_{sb} = [\mathcal{V}_s] T_{sb}$

$$= \begin{bmatrix} e^{[\hat{\omega}]\theta} & G(\theta)v \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{[\omega_s]\theta} & G(\theta) v_s \\ 0 \; 0 \; 0 & 1 \end{bmatrix}$$

$$G(\theta) = I\theta + [\omega]\frac{\theta^2}{2!} + \cdots = I\theta + (1 - \cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2$$

Logarithm of rigid body motions

Given $T(R, p) \in SE(3)$, find $\mathcal{S} = [\omega \quad v]^{\mathsf{T}}$ and θ such that:

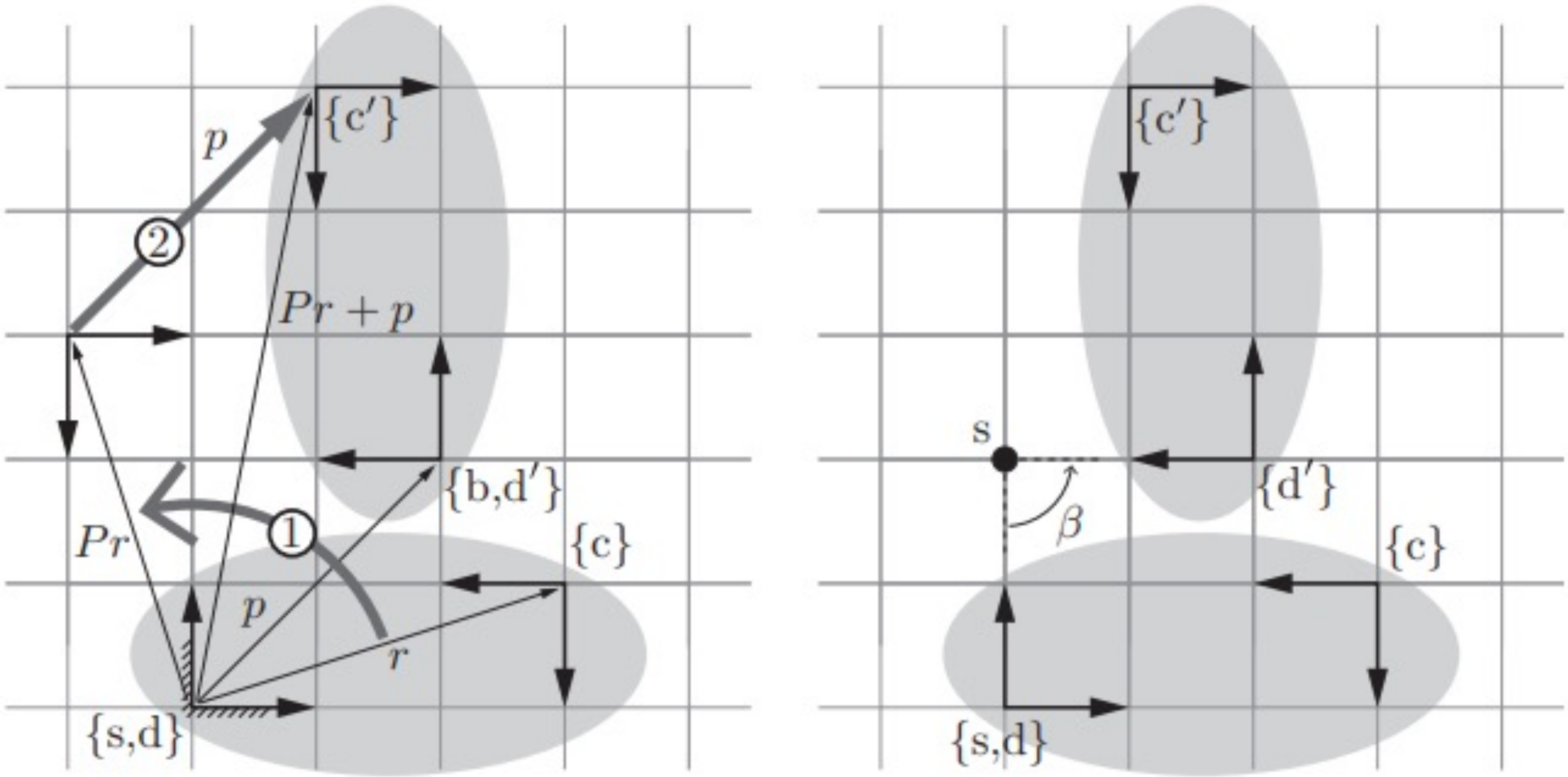
$$e^{[\mathcal{S}]\theta} = \boxed{\begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}} \rightarrow [\mathcal{S}]\theta = \begin{bmatrix} [\omega]\theta & v\theta \\ 0 & 0 \end{bmatrix}$$

Define the logarithm of T

1. If $R = I$, then $\omega = 0$ and $v = \frac{p}{\|p\|}$ and $\theta = \|p\|$
2. Find ω, θ using matrix logarithm of R (see earlier slide!)
   Then, $v = G^{-1}(\dot{\theta})p$, where $G^{-1}(\dot{\theta}) = \frac{1}{\theta} I - \frac{1}{2}[\omega] + \left(\frac{1}{\theta} - \frac{1}{2}\cot\left(\frac{\theta}{2}\right)\right)[\omega]^2$

Now we can go back and forth between exponential coordinates and transformations!

Screw Motions



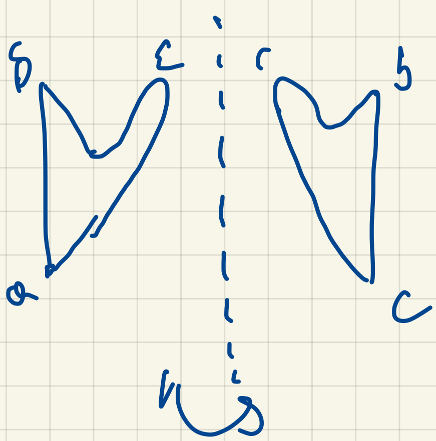
- What are all possible rigid transformations* in 2D & 3D

* orientation preserving.

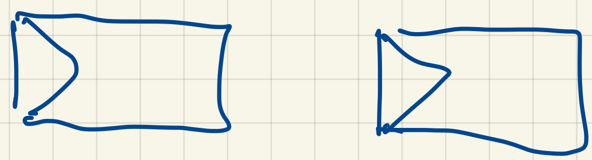
- Chasles – Mozzi
- Ball.

- rigid : preserves lengths & angles.

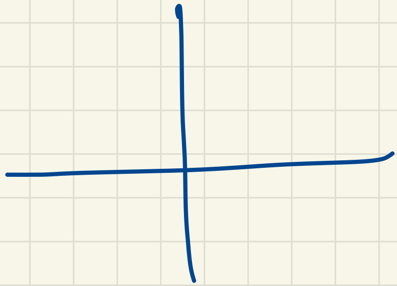
orientation preserving :

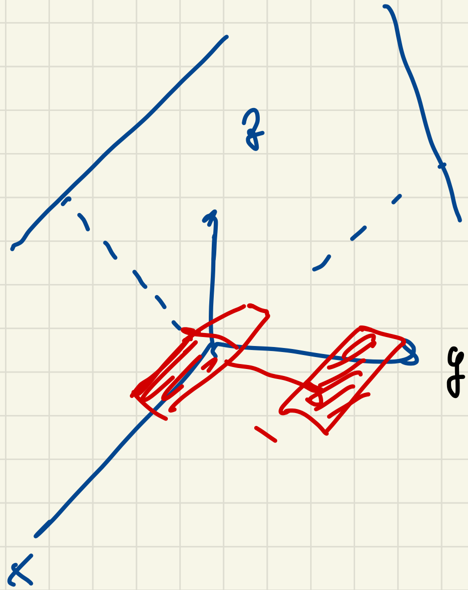


reversal of orientation.



(Δ_x, δ_y)





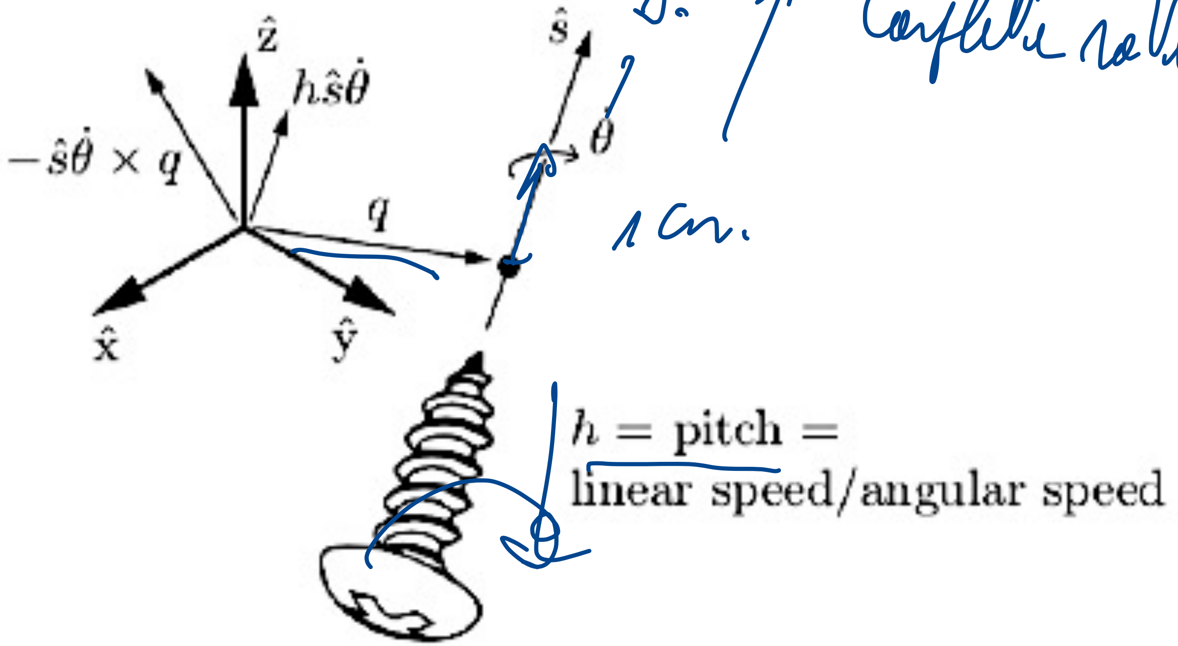
2D Screw Motions

- Any rigid motion in the plane can be represented by a rotation around a well-chosen center
- We can encode it with (β, s_x, s_y), where (s_x, s_y) is the position of the center of the rotation, and β the angle

- Recall that the angular velocity ω can be viewed as $\hat{\omega}\dot{\theta}$, where $\hat{\omega}$ is the unit rotation axis and $\dot{\theta}$ is the rate of rotation
- A twist $\mathcal{V}$ can be interpreted in terms of a screw axis $\mathcal{S}$ and a velocity $\dot{\theta}$ about that axis

Screw motions in 3D

h units for a complete rotation



Chasles-Mozzi Theorem:
Any displacement in 3D can be represented by a rotation and translation about the same axis, referred to as a screw motion

$q \in \mathbb{R}^3$ is any point along the axis of the screw motion.

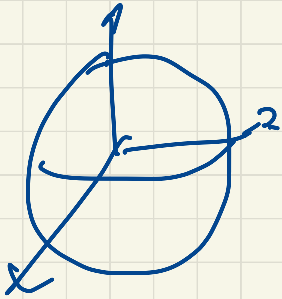
$\hat{s}$ is a unit vector in the direction of the axis

h is the **screw pitch**, which is the ratio between the linear and angular speed along axis

We write this as $\mathcal{S} = \{q, \hat{s}, h\}$

(note: we need 6 independent parameters for S)

$\hat{s}$ is normalized

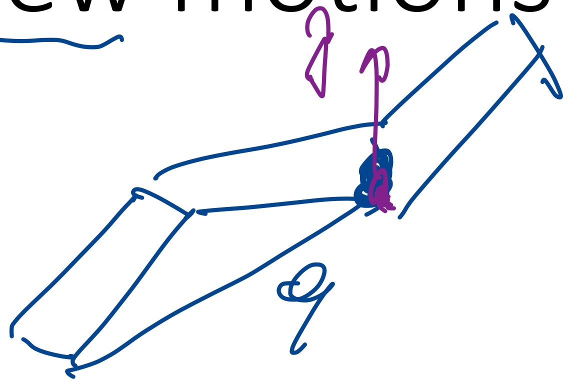


$$\hat{s} = \begin{pmatrix} s_x \\ s_y \\ s_z \end{pmatrix}$$

$$s_x^2 + s_y^2 + s_z^2 = 1$$

$\sim$ 2 DOF.

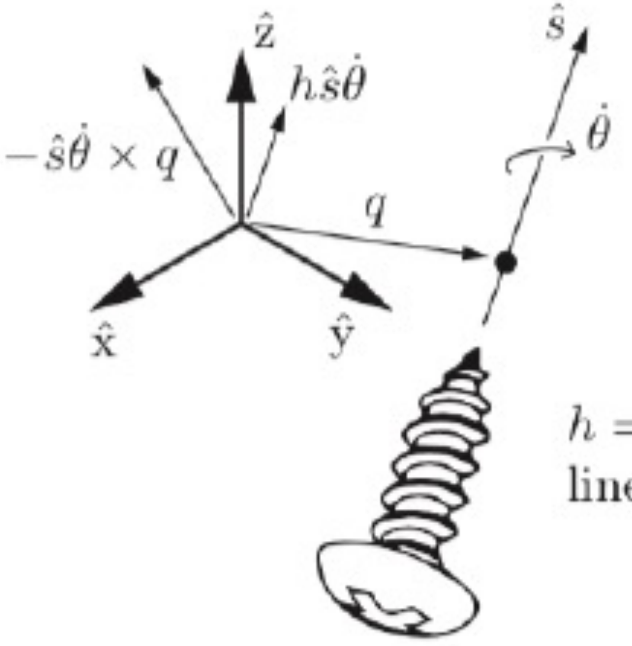
Screw motions and Twists



$q, \hat{s}, h$

4

0.



$$\{q, \hat{s}, h\} \rightsquigarrow \mathcal{V}_s = \begin{pmatrix} \omega_s \\ v_s \end{pmatrix}.$$

$$\begin{cases} \omega_s = \hat{s} \\ v_s = -\hat{s}\dot{\theta} \times q + h\hat{s}\dot{\theta} \end{cases}$$

Screw axis

$$\mathcal{V}_s \rightsquigarrow \{q, \hat{s}, h\}$$

$$\omega_s = \hat{s}$$

$$v_s = -q \times \hat{s}\dot{\theta} + h\hat{s}\dot{\theta}$$

$$\begin{pmatrix} \omega_s \\ v_s \end{pmatrix} \qquad \hat{s} = \omega_s / \|\omega_s\| , \quad \dot{\theta} = \|\omega_s\|$$

note that $\hat{s}^T v_s = 0 + h\dot{\theta}$

$$\rightsquigarrow h = \frac{\hat{s}^T v_s}{\|\omega_s\|}$$

$$q = ? \quad \hat{s} \times v_s = \hat{s} \times (-\hat{s}\dot{\theta} \times q) + \underbrace{\cancel{\hat{s} \times (h\hat{s}\dot{\theta})}}_{0}$$

$$= \dot{\theta} q \,]$$

Summary

- Used transformations to define spatial velocities as **body and spatial twists**
- Introduced **screw motions** and the screw interpretation of a twist
- The **screw axis $\mathcal{S}$** was defined and used to define the **exponential coordinates of homogeneous transformations**
- Connections between last lecture and this lecture:
  - Transformation matrix T is analogous to rotation matrix R
  - A screw axis $\mathcal{S}$ is analogous to rotation axis $\widehat{\omega}$
  - A twist $\mathcal{V}$ can be expressed as $\mathcal{S}\dot{\theta}$ and is analogous to angular velocity $\omega = -\widehat{\omega}\dot{\theta}$
  - Exponential coordinates $\mathcal{S}\theta \in \mathbb{R}^6$ for rigid body motions are analogous to exponential coordinates $\widehat{\omega}\theta \in \mathbb{R}^3$

09/25 : - HW 4 is posted.
(use it to practice FR!)

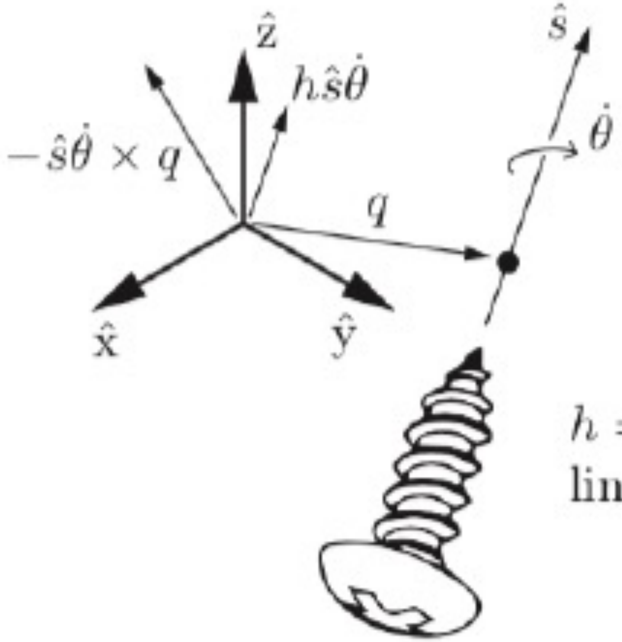
- PE1 is also posted

Quick Recap of Past Lectures

1 We can define rotation matrices with a matrix exponential: $R = e^{[\hat{\omega}]\theta}$
Where $e^{[\hat{\omega}]\theta}$ is computed with Rogrigues's Formula
$\hat{\omega}\theta$ are referred to as exponential coordinates

2 For rotations, the only velocities are angular.
For transformations (rotations + translations), we need angular and linear (spatial) velocities!
Twists are the concatenation of ω and v

3 Screw motions give us a generalized concept for defining any motion with a direction and a "screw".
From the information in a twist, we can compute all the components for a screw motion.
Can we connect screws with exponential coordinates to give us a general equation for T?



Screw axis

Given a ref frame, a screw axis $\mathcal{S}$ is defined:

$$\mathcal{S} = \begin{bmatrix} \omega \\ v \end{bmatrix} \in \mathbb{R}^6$$

note: these are overloaded vars!
for screw motions, either $\|\omega\|$ or $\|v\|$ must be 1
→ for twists, ω + v are unconstrained

where either

① $\|\omega\| = 1$

$v = -\omega \times q + h\omega$

h=0 gives pure rot

or ② $\|v\| = 1$ and $\omega = 0$

→ translating along v

$$[\mathcal{S}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \in se(3), \quad \mathcal{S}_a = [Ad_{T_{ab}}]\mathcal{S}_b$$

Example of Screw Motion for Transformations

Find the pose that is produced by a given screw

Frame 0 is fixed in space and frame 1 is fixed in a body. The current pose of frame 1 in the coordinates of frame 0 is M. The body undergoes a body screw motion with axis $\mathcal{B}$ and magnitude θ. Find the new pose T_{01} of frame 1 in the coordinates of frame 0.

matlab python

```
M = [0.03262475 -0.39
B = [0.00000000; 0.00
theta = 1.29638080;
```

copy this text

[1]:

```
M = [0.03262475 -0.39471442 0.91822445 -0.58853212; -0.21629944 -0.89972977 -0.37907900
B = [0.00000000; 0.00000000; 0.00000000; -0.67130813; -0.02965809; 0.74058477];
theta = 1.29638080;

bb=[0 -B[3] B[2] B[4];
    B[3] 0 -B[1] B[5];
    -B[2] B[1] 0 B[6];
    0 0 0 0];
tt=exp(bb*theta);
T=M*tt
```

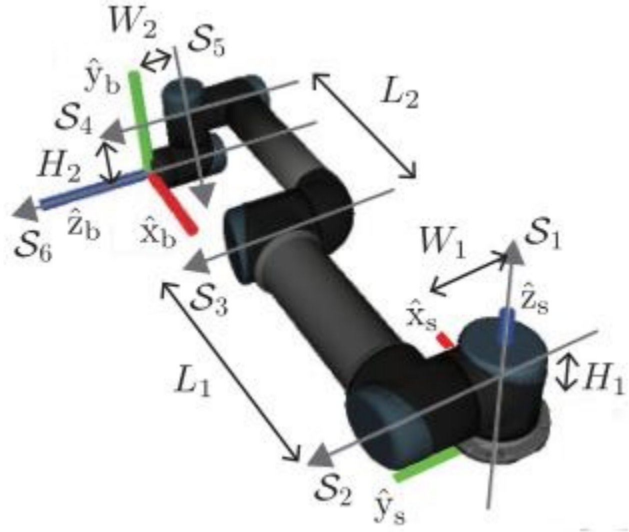
[1]:

```
4×4 Array{Float64,2}:
  0.0326248  -0.394714   0.918224   0.27982
 -0.216299   -0.89973   -0.379079   0.61705
  0.975782   -0.186244  -0.11473   -0.285795
  0.0         0.0        0.0        1.0
```

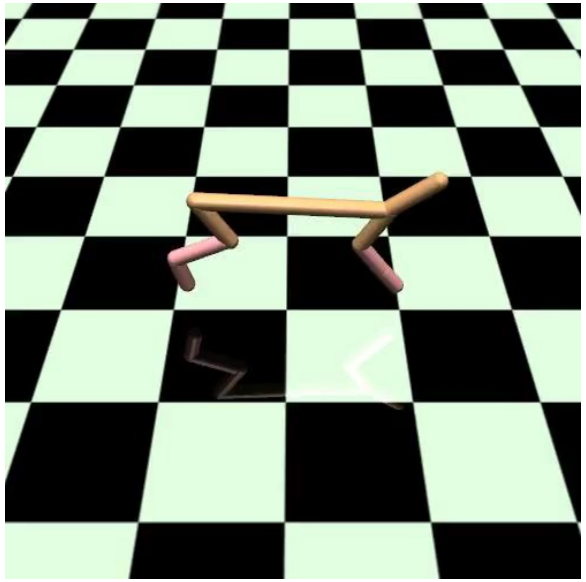
On to Forward Kinematics!

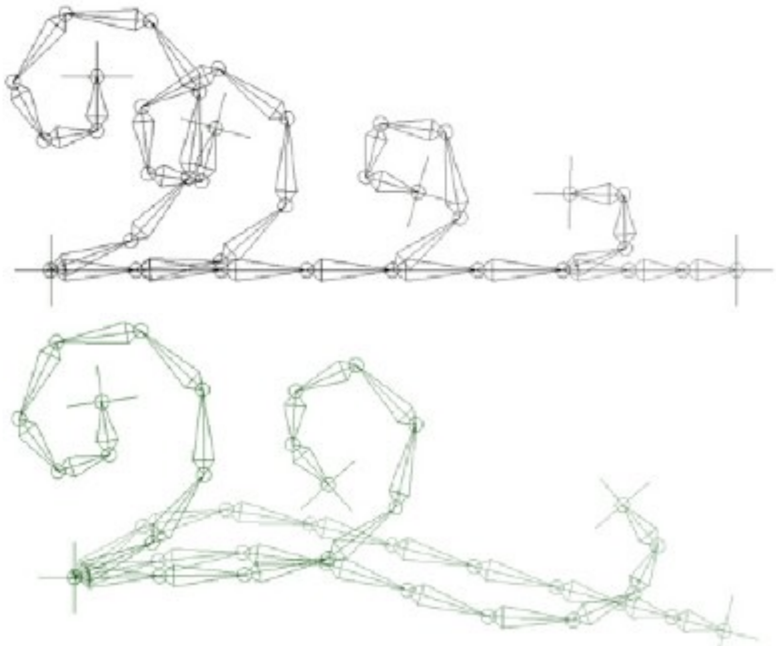
What is Forward Kinematics?

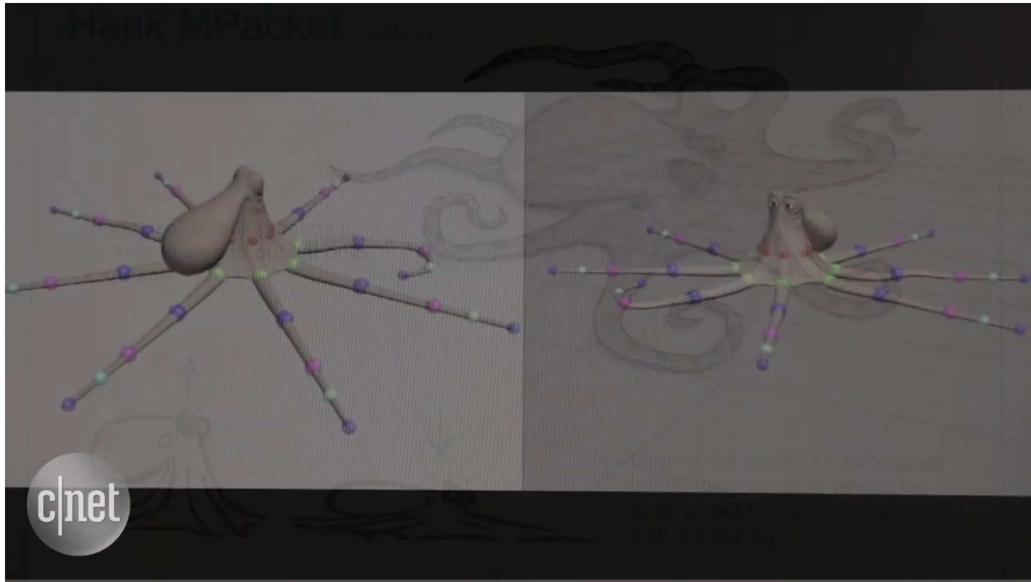
- **Kinematics:** a branch of classical mechanics that describes motion of bodies without considering forces. AKA “the geometry of motion”
- **Forward kinematics:** a specific problem in robotics. Given the individual state of each joint of the robot (in a local frame), what is the position of a given point on the robot in the global frame?



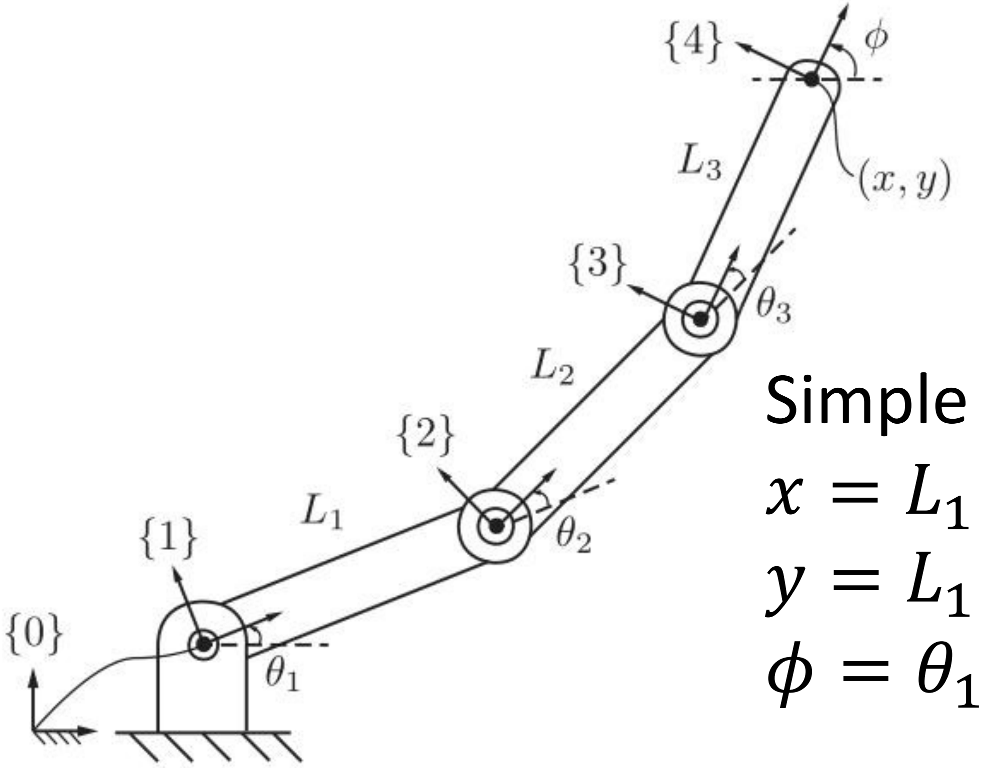
Where is forward kinematics used?







Forward Kinematics of a Simple Chain



Simple Geometry!

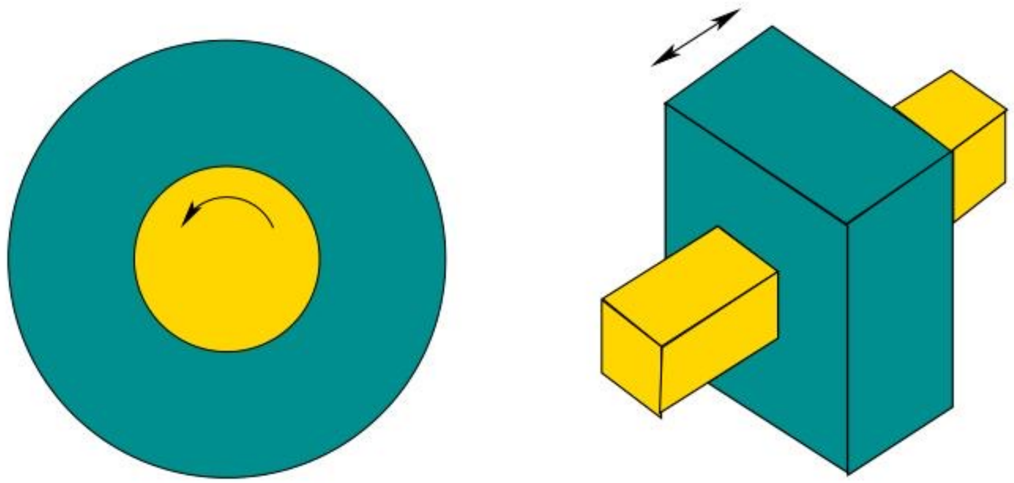
$$x = L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) + L_3 \cos(\theta_1 + \theta_2 + \theta_3)$$
$$y = L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2) + L_3 \sin(\theta_1 + \theta_2 + \theta_3)$$
$$\phi = \theta_1 + \theta_2 + \theta_3$$

Can we use transformations instead?

$$T_{04} = T_{01}T_{12}T_{23}T_{34}$$
$$T_{sb} = T_{s1}T_{12} \cdots T_{n-1,n}M$$

Assumptions

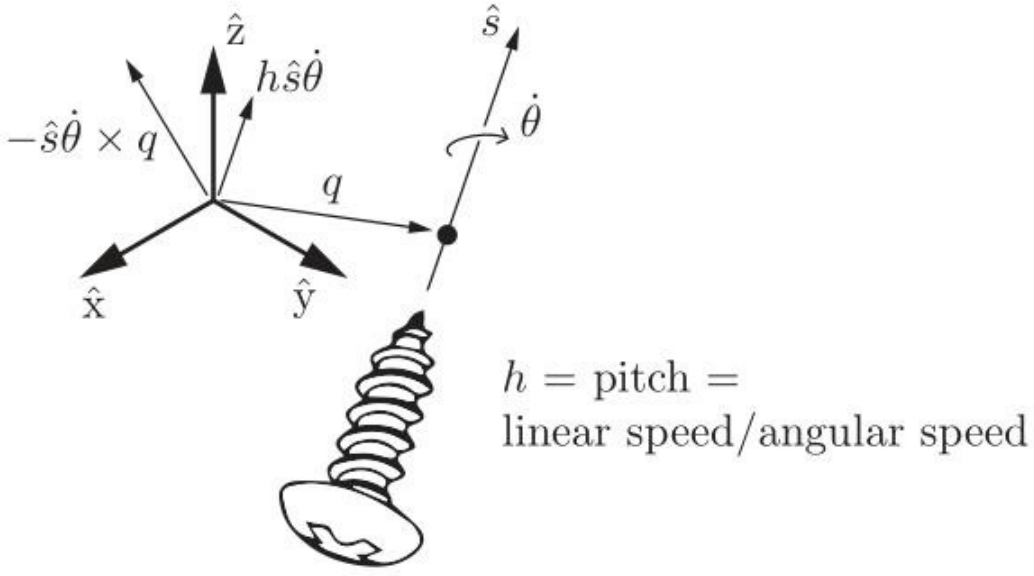
- Our robot is a kinematic chain, made of rigid links and movable joints
- No branches or loops
- All joints have one degree of freedom and are revolute or prismatic



Revolute
1 Degree of Freedom

Prismatic
1 Degree of Freedom

Review of Screw Motions



We derived a representation for the screw axis:

$$\mathcal{S} = \begin{bmatrix} \omega \\ v \end{bmatrix} \in \mathbb{R}^6$$

where either

- $\|\omega\| = 1$
  - where $v = -\omega \times q + h\omega$, where q is the point on the axis of the screw and h is the pitch of the screw
- $\|\omega\| = 0$ and $\|v\| = 1$

Screw Motions as Matrix Exponential

The screw axis $\mathcal{S}_i$ can be expressed in matrix form as:

$$[\mathcal{S}_i] = \begin{bmatrix} [\omega_i] & v \\ 0 & 0 \end{bmatrix} \in se(3)$$

To express a **screw motion** given a screw axis, we use the matrix exponential:

$$e^{[\mathcal{S}]\theta} \in SE(3)$$

Proposition 3.25. *Let $\mathcal{S} = (\omega, v)$ be a screw axis. If $\|\omega\| = 1$ then, for any distance $\theta \in \mathbb{R}$ traveled along the axis,*

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} e^{[\omega]\theta} & \left(I\theta + (1-\cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2\right) v \\ 0 & 1 \end{bmatrix}. \tag{3.88}$$

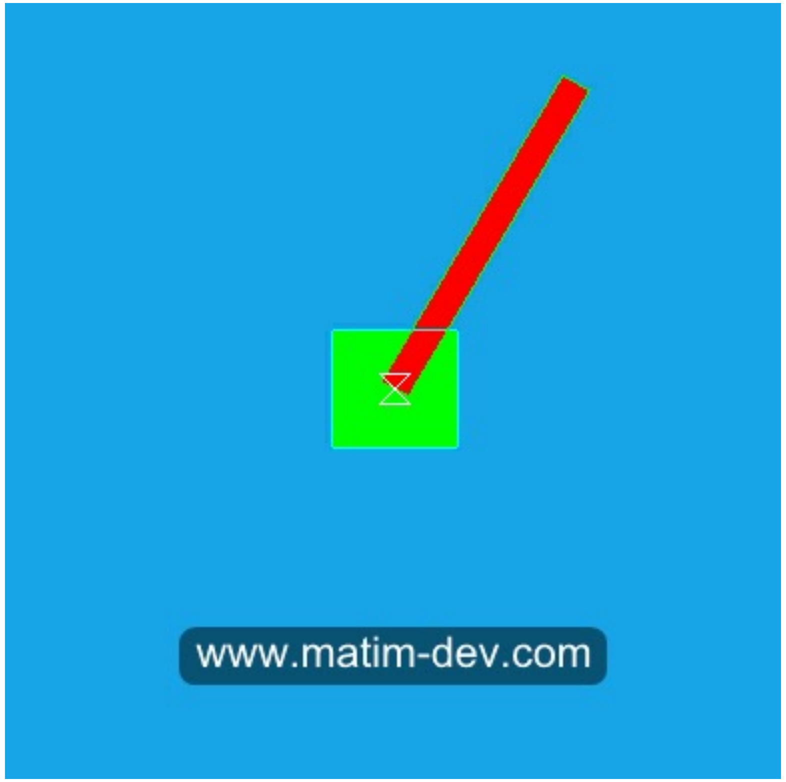
If $\omega = 0$ and $\|v\| = 1$, then

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix}.$$

Modeling Robot Joints as Screw Motions

Case 1: Revolute Joint

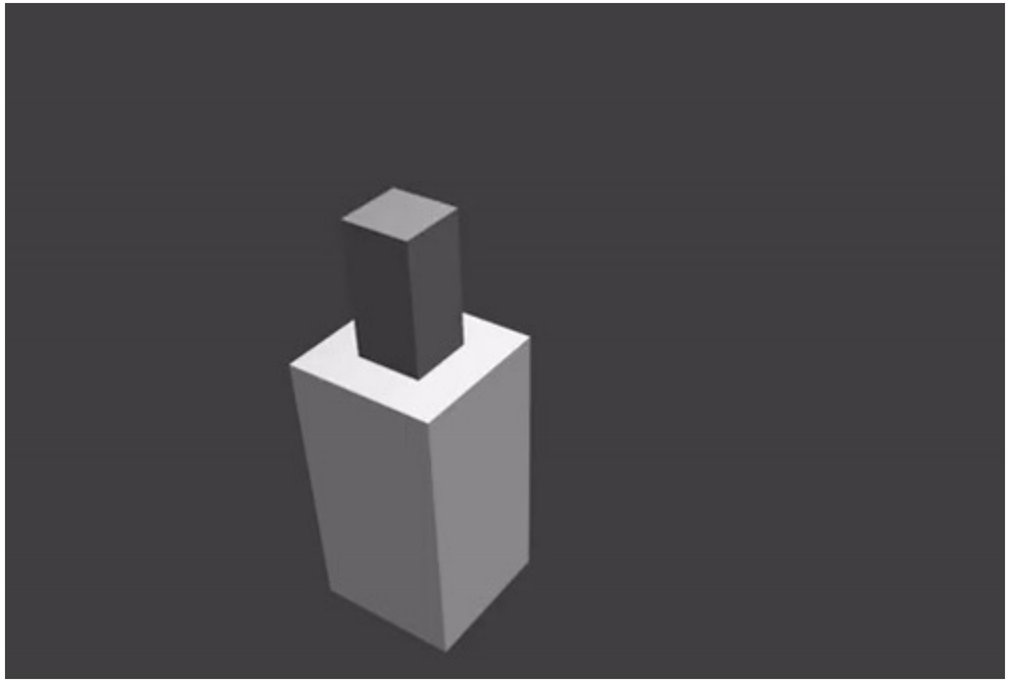
- $\|\omega\| = 1$
- $v = -\omega \times q + \cancel{h\omega}$
- $v = -\omega \times q$



Modeling Robot Joints as Screw Motions

Case 2: Prismatic Joint

- $||\omega|| = 0$
- $||v|| = 1$
- Axis of movement defines v



Product of Exponentials Approach

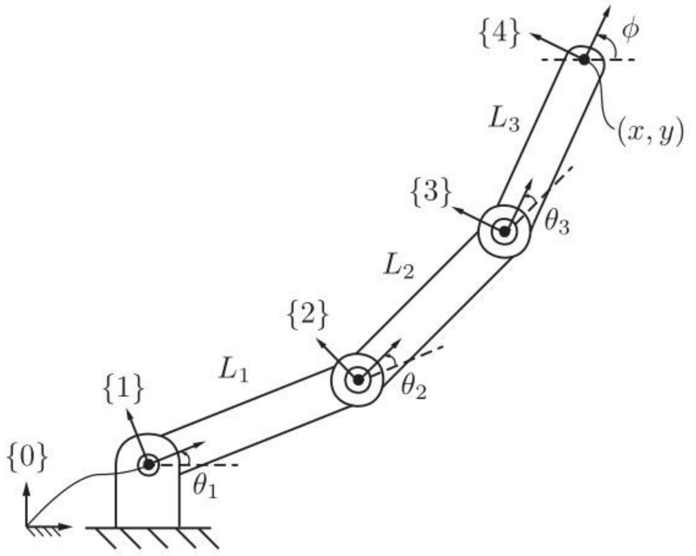


Figure 4.1: Forward kinematics of a 3R planar open chain. For each frame, the x̂- and ŷ-axis is shown; the ẑ-axes are parallel and out of the page.

Let each joint i have a configuration defined by θ_i

Initialization steps:

- Choose a fixed frame $\{s\}$
- Choose an end-effector (tool) frame attached to the robot $\{b\}$
- Put all joints in *zero position*
- Let $M \in SE(3)$ be the configuration of $\{b\}$ in the $\{s\}$ frame when the robot is in the zero position

Product of Exponentials Approach

- Given zero position M
- For each joint i, define the screw axis
- For each motion of a joint, define the screw motion
- These operations compose nicely through multiplication, giving us the Product of Exponentials (PoE) formula!

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} \dots e^{[\mathcal{S}_{n-1}]\theta_{n-1}} e^{[\mathcal{S}_n]\theta_n} M$$

Visualizing the Product of Exponentials

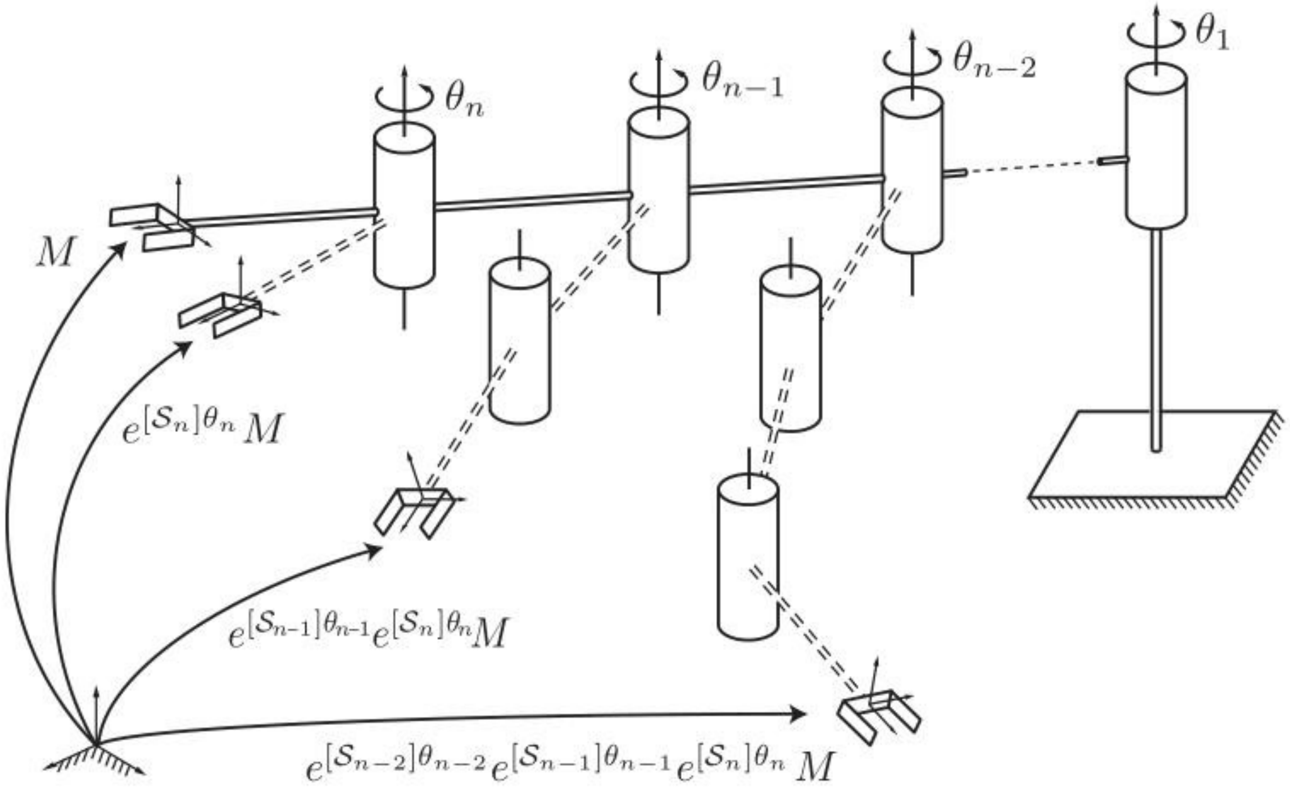


Figure 4.2: Illustration of the PoE formula for an n-link spatial open chain.

Example 1

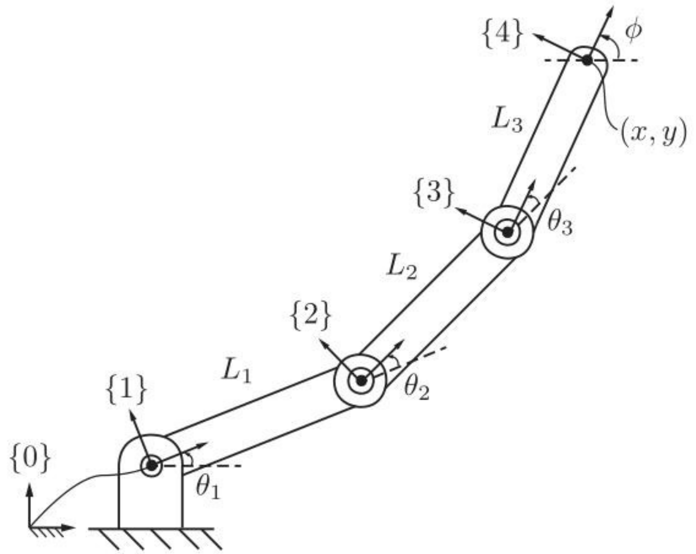


Figure 4.1: Forward kinematics of a 3R planar open chain. For each frame, the x̂- and ŷ-axis is shown; the ẑ-axes are parallel and out of the page.

Example 1: PoE

Compute $e^{[\mathcal{S}_i]\theta_i}$ for each joint:

$$e^{[\mathcal{S}_i]\theta_i} = \begin{bmatrix} e^{[\omega_i]\theta_i} & (I\theta_i + (1 - \cos\theta_i)[\omega_i] + (\theta_i - \sin\theta_i)[\omega_i]^2)v_i \\ 0 & 1 \end{bmatrix}$$

and compose with M:

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M$$

Example 2

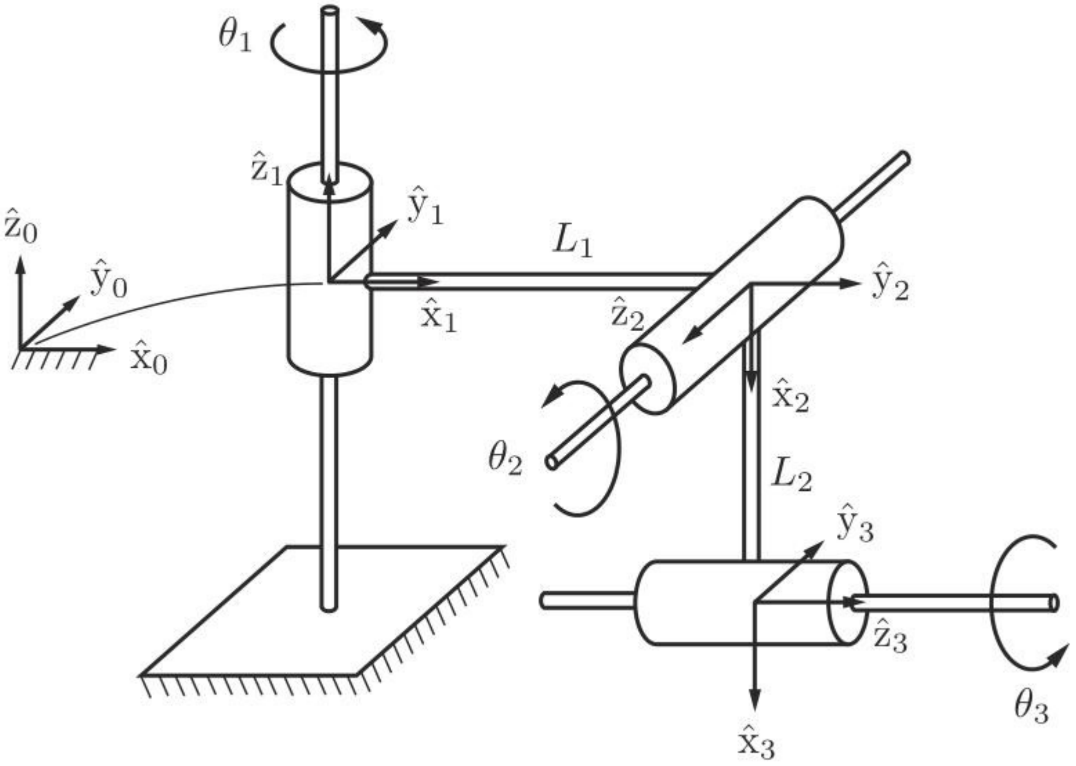


Figure 4.3: A 3R spatial open chain.

Example 3 – How to read this diagram

RPR

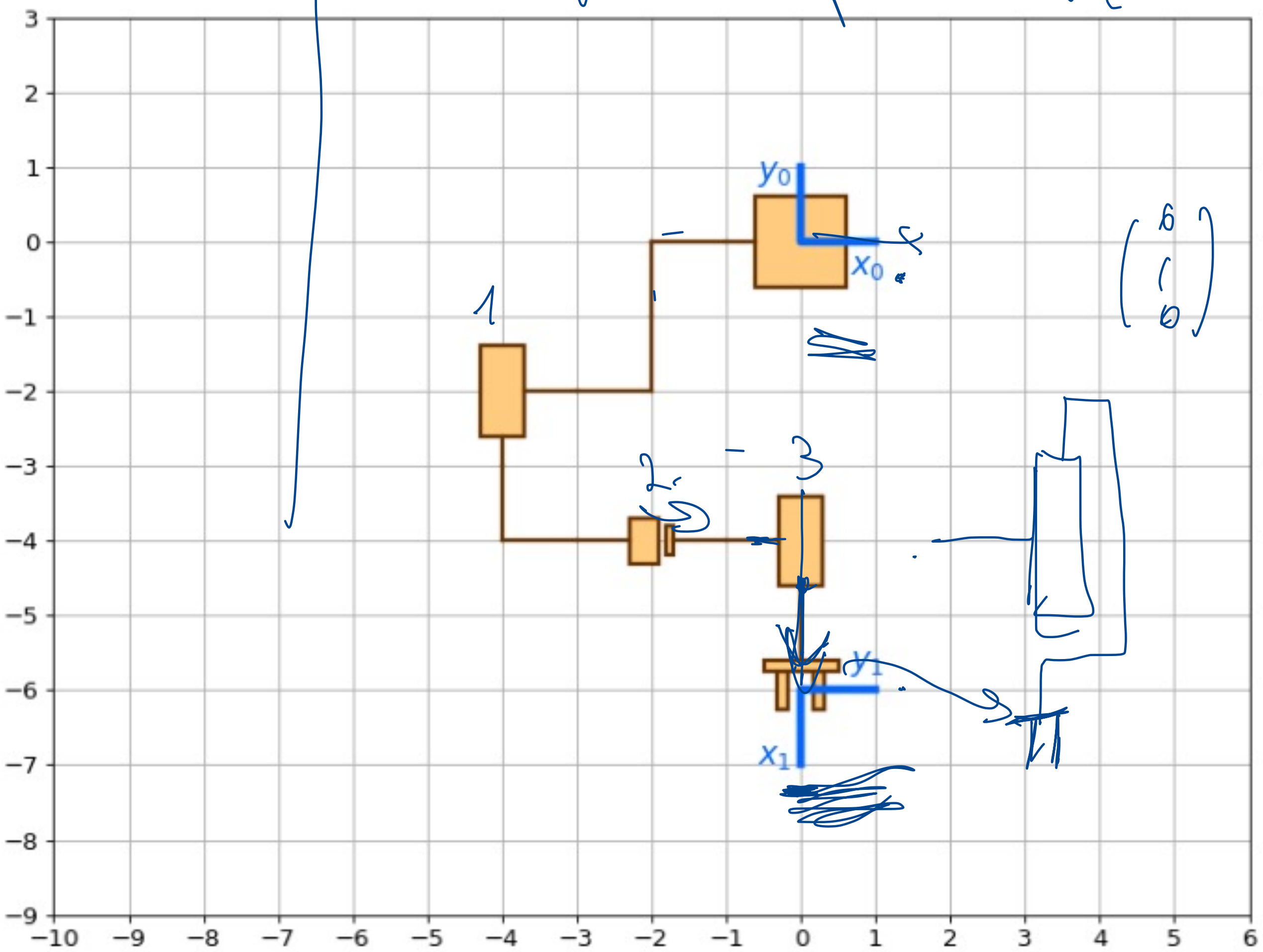
$\omega = 0$

$v = $ axis of translation

$V_i = -\omega_i \times q_i$

Revolut

$$M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



ω_1: axis ω $\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$

$S_1 = \begin{pmatrix} 0 \\ -1 \\ 0 \\ v_1 \end{pmatrix}$ $\quad S_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ $\quad S_3 = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$q_1 = \begin{pmatrix} -4 \\ 0 \\ 0 \end{pmatrix}$ $\quad v_1 = -\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \times \begin{pmatrix} -4 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$

Example 3

v_i , ω_i normalized

$[S_1 \; S_2 \; \cdots \; S_6]$

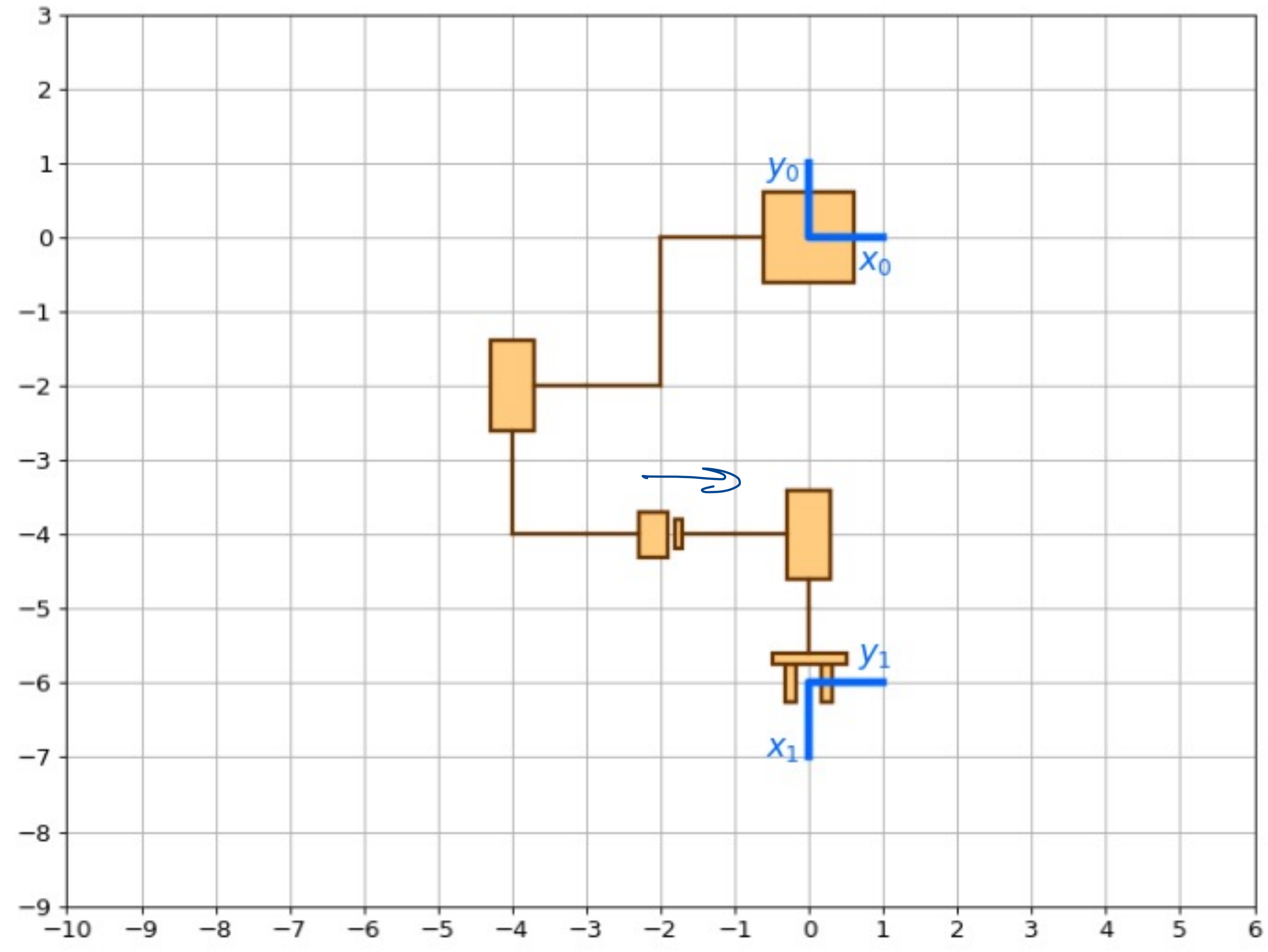
$e^{[S_i]\theta_i}$

or $e^{[S_i]\theta_i}$

$v_i = -\omega_i \times q_i$

$$S_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$e^{[S_2]\theta_2}$



PoE in the End-Effector Frame

- Recall some identities:

$$e^{M^{-1}PM} = M^{-1}e^{P}M \rightarrow e^{P}M = Me^{M^{-1}PM}$$

- Applying this to PoE:

$T(\theta) = e^{[\mathcal{S}_1]\theta_1} \dots e^{[\mathcal{S}_n]\theta_n} M$

$T(\theta) = e^{[\mathcal{S}_1]\theta_1} \dots Me^{M^{-1}[\mathcal{S}_n]M\theta_n}$

$T(\theta) = e^{[\mathcal{S}_1]\theta_1} \dots Me^{M^{-1}[\mathcal{S}_{n-1}]M\theta_{n-1}}e^{M^{-1}[\mathcal{S}_n]M\theta_n}$

$T(\theta) = Me^{M^{-1}[\mathcal{S}_1]M\theta_1} \dots e^{M^{-1}[\mathcal{S}_{n-1}]M\theta_{n-1}}e^{M^{-1}[\mathcal{S}_n]M\theta_n}$

- What is $M^{-1}[\mathcal{S}_i]M$ (or $[Ad_{M^{-1}}]\mathcal{S}_i$)?
  - Screw axis in the body frame: $[\mathcal{B}_i]$
- Body form of PoE:

$$T(\theta) = Me^{[\mathcal{B}_1]\theta_1} \dots e^{[\mathcal{B}_{n-1}]\theta_{n-1}}e^{[\mathcal{B}_n]\theta_n}$$

Summary

- Used transformations to define spatial velocities as the **body and spatial twist**
- Introduced **screw motions** and the screw interpretation of a twist
- The **screw axis** $\mathcal{S}$ was defined and used to define the **exponential coordinates of homogeneous transformations**
- Learned the basics of **forward kinematics**, which gives us a model for computing position and orientation of the end-effector
- Uses the **product of exponentials formula** to define this transformation as composed matrix multiplications

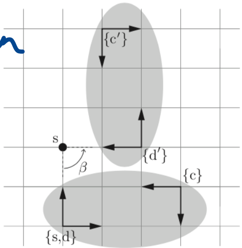
Introduction to Robotics
Screws and wrenches

MA Belabbas

Screw motion

All* motions in 2D are rotations (about a well-chosen center)

* rigid, orientation preserving.

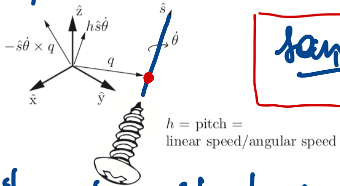


- 2D Screw motion: any rigid motion in the plane can be represented by a rotation around a well-chosen center.
- We can encode it with (β, s_x, s_y), where (s_x, s_y) is the position of the center of the rotation, and β the angle. Here, $\beta = \pi/4$ and $s = (0, 2)$

Screw motion

in 3D: rotation & translation about the same axis.

note: by convention, the translation amount is given by $\theta . h$

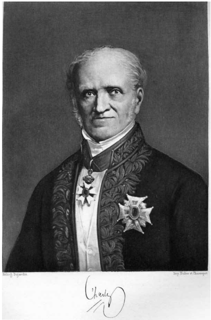


(eg, if we translate by 10 cm & rotate by 10°, then $h = 1$ cm/deg, $\theta = 10°$

- **Chasles-Mozzi Theorem**: Any displacement in 3D can be represented by a rotation + translation about same axis. This is called a screw motion.
- Data for a screw: Axis of rotation: $q \in \mathbb{R}^3 + \hat{s} \in S^2$; pitch h : ratio linear/angular speed; $\dot{\theta}$: angular velocity.
- Rotation of θ rad results in translation of $h\theta$ along axis.
- Very useful to represent motion of revolute and prismatic joints.
- Write as $\mathcal{S} = \{q, \hat{s}, h\}$. 6 parameters; well suited to describe joints

Ball and Chasles

Chasles



- Rodrigues and Chasles took the entrance exam to Polytechnique/Normale at the same time, finishing first and second respectively. Rodrigues did not use it and elected to go to La Sorbonne.

Screw motion → twist $\{q, \hat{s}, h\}$

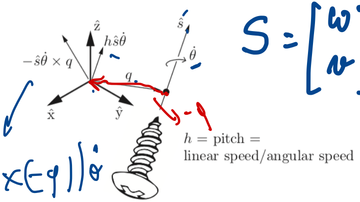
recall: $\dot{x} = \omega \times x$

$\omega \times x$

$$S = \begin{bmatrix} \omega \\ v \end{bmatrix}$$

$\omega_s = \hat{s}$

$v_s = (-\hat{s} \times q)\dot{\theta} + h \hat{s} \dot{\theta}$



$(\hat{s} \times (-q))\dot{\theta}$

- How to represent a screw motion as a twist? Take $\mathcal{S} = \{q, \hat{s}, h\}$.
  - Rotational velocity: $\omega = \hat{s}\dot{\theta}$.
  - Linear velocity : velocity at origin of frame= translation along the screw axis + linear motion at origin induced by rotation of screw axis:

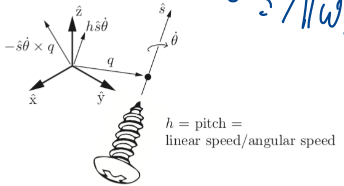
$$v = h\hat{s}\dot{\theta} - \hat{s}\dot{\theta} \times q.$$

Twist → Screw motion

$V_s = -(\hat{s} \times q)\dot{\theta} + h\hat{s}\dot{\theta}$

$\begin{pmatrix}\omega_s \\ v_s\end{pmatrix} \overset{?}{\to} (q, \hat{s}, h)$

$\omega_s / \|\omega_s\| = \hat{s}$



- How to represent a twist as a screw motion? Take $\mathcal{V} = [\omega, v]^\top$
  - Rotation+pitch: $\hat{s} = \omega/\|\omega\| = \bar{\omega}$, $\dot{\theta} = \|\omega\|$ and $h = \bar{\omega}^\top v/\dot{\theta} = \omega^\top v/\|\omega\|^2$.
  - Offset q: Assuming $\omega \neq 0$ (not pure translation, so that there is a meaningful rotation axis), need to find q so that $v = h\hat{s}\dot{\theta} - \hat{s}\dot{\theta} \times q$. Cross product on left by $\hat{s}$ yields

$$q = \frac{\hat{s} \times v}{\dot{\theta}}.$$

$$v_s = -(\hat{s} \times q)\dot{\theta} + h\,\hat{s}\,\dot{\theta}$$

recall $\hat{s}^T v_s = \langle \hat{s}, v_s \rangle = \underbrace{\hat{s} \cdot v_s}_{\text{inner or dot product}}$

but: $\hat{s} \times q$ is by definition $\perp$ to both q & $\hat{s}$!!

$$\Rightarrow \underset{(-\cdots)}{\hat{s}^T} \underset{\left(\begin{smallmatrix}\vdots\end{smallmatrix}\right)}{v_s} = h \underbrace{\hat{s}^T \hat{s}}_{\|\hat{s}\|^2 = 1} \dot{\theta} = h\dot{\theta}$$

$$\rightarrow \quad h = \frac{\hat{s}^T v_s}{\dot{\theta}} = \frac{\left[\omega_s / \|\omega_s\|\right]^T v_s}{\|\omega_s\|}$$

$$= \frac{\omega_s^T v_s}{\|\omega_s\|^2}$$

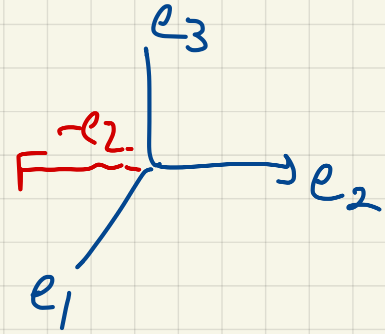
$$\begin{pmatrix} \omega_s \\ v_s \end{pmatrix} \rightsquigarrow \quad \begin{aligned} \hat{s} &= \omega_s / \|\omega_s\| \\ \dot{\theta} &= \|\omega_s\| \\ h &= \frac{\omega_s^T v_s}{\|\omega_s\|^2} \end{aligned}$$

How about q?

$$v_s = -(\hat{s} \times q)\dot{\theta} + h \hat{s} \dot{\theta}$$

$$\hat{s} \times v_s = \hat{s} \times (-\hat{s} \times q)\dot{\theta} + \underbrace{\hat{s} \times h \hat{s} \dot{\theta}}_{0}$$

example: $\underbrace{e_1 \times \underbrace{(e_1 \times e_2)}_{e_3}}_{-e_2}$



$$= -(\hat{s} \times (\hat{s} \times q))\dot{\theta} + 0$$

$$= -(-q)\dot{\theta}$$

$$= q\dot{\theta}$$

$$\Rightarrow \quad q = (\hat{s} \times v_s)/\|\omega_s\|$$

$\therefore$ To formally show that

$\hat{s} \times (\hat{s} \times q) = -q$ for any normalized $\hat{s}$ non // to q, we determinant formula.

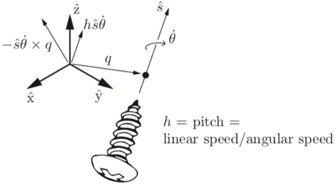
$$\begin{pmatrix} \omega_s \\ v_s \end{pmatrix} \rightarrow \quad \hat{s} = \omega_s / \|\omega_s\|$$

$$\dot{\theta} = \|\omega_s\|$$

$$h = \frac{\omega_s^T v_s}{\|\omega_s\|^2}$$

$$q = \frac{\omega_s \times v_s}{\|\omega_s\|^2}$$

Screw motion



- For a given reference frame, we denote by $\mathcal{S}$ the *screw axis* of the screw motion $\{q, \hat{s}, \dot{\theta}\}$

$$\mathcal{S} = \begin{bmatrix} \omega \\ v \end{bmatrix}$$

$S = \begin{pmatrix} 0 \\ v \end{pmatrix}$ $\therefore$

with either 1: $\|\omega\| = 1$ or 2: $\omega = 0$ and $\|v\| = 1$.
Case 1: then $v = -\omega \times q + h\omega$
Case 2: corresponds to a translation along v: (h is infinite) $\rightarrow \omega = 0$.

Exponential coordinates for Rigid-Body motions

- Recall: for a rotation $R \in SO(3)$, its *exponential coordinates* $(\bar{\omega}, \theta)$ are so that $R = e^{[\bar{\omega}]\theta}$. We want to do the same for rigid motions (i.e., rotation + translation)
- The matrix $[\bar{\omega}]$ was skew-symmetric: $[\bar{\omega}] \in \mathfrak{so}(3)$.
- We defined

$$\exp : \mathfrak{so}(3) \to SO(3) : A \to I + A + A^2/2! + \cdots$$
$$\log : SO(3) \to \mathfrak{so}(3) : R \to \frac{1}{2\sin\theta}(R - R^\top)$$

Exponential coordinates for Rigid-Body motions

- By analogy, we define the exponential coordinates for a homogeneous transformation T as $\mathcal{S}\theta \in \mathbb{R}^6$, where $\mathcal{S}$ is the screw axis and θ the distance around the screw axis to take a frame from I to T.
- Recall that if $\mathcal{S} = (\omega, v)$ is with $\|\omega\| = 1$, then θ is an angle of rotation about the screw axis. If $\omega = 0$ and $\|v\| = 1$, then θ is the distance travelled along the screw axis.
- Denote by $SE(3)$ the space of homogeneous transformations T, and by $\mathfrak{se}(3)$ the space of their exponential coordinates. We want to define

$$\begin{aligned}
\exp :& \mathfrak{se}(3) \to SE(3) : A \to I + A + A^2/2! + \cdots \\
\log :& SE(3) \to \mathfrak{se}(3) : T \to [\mathcal{S}]\theta \in \mathfrak{se}(3)
\end{aligned}$$

Exponential coordinates for Rigid-Body motions

For $\mathcal{S} = [\omega, v]^\top$, we introduce the 4×4 matrix:

$$[\mathcal{S}] := \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix}$$

- As before, we can find a simple expression for the exponential. We have (recall: $\omega = \overline{\omega}\theta$)

$$\begin{aligned} e^{[\mathcal{S}]\theta} &= I + [\mathcal{S}]\theta + [\mathcal{S}]^2\theta^2/2! + [\mathcal{S}]^3\theta^3/3! + \cdots \\ &= \begin{bmatrix} e^{[\overline{\omega}]\theta} & G(\theta)v \\ 0 & 1 \end{bmatrix} \rightsquigarrow \text{computed fast.} \end{aligned}$$

  where $G(\theta) = I\theta + [\overline{\omega}]\theta^2/2! + [\overline{\omega}]^2\theta^3/3! + \cdots$.
- Recall that $[\overline{\omega}]^3 = -[\overline{\omega}]$. We obtain

$$\begin{aligned} G(\theta) &= I\theta + (\theta^2/2! - \theta^4/4! + \cdots)[\overline{\omega}] + (\theta^3/3! - \theta^5/5! + + \cdots)[\overline{\omega}]^2 \\ &= I\theta + (1 - \cos\theta)[\overline{\omega}] + (\theta - \sin\theta)[\overline{\omega}]^2 \end{aligned}$$

Logarithm of Rigid-Body motions

- Given $T = (R, p) \in SE(3)$, we need to find $\mathcal{S} = (\overline{\omega}, v)$ and θ so that

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix},$$

  which implies that

$$[\mathcal{S}]\theta = \begin{bmatrix} [\overline{\omega}]\theta & v\theta \\ 0 & 0 \end{bmatrix}$$

  is a logarithm of T.
- Case 1: If $R = I$ (pure translation), then set $\overline{\omega} = 0$, $v = p/\|p\|$ and $\theta = \|p\|$.
- Case 2: **1.** Obtain $\overline{\omega}, \theta$ using the matrix logarithm of R. **2.** Set

$$v = G^{-1}(\theta)p$$

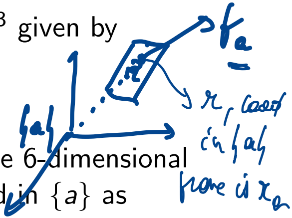
  where

$$G^{-1}(\theta) = \frac{1}{\theta}I - \frac{1}{2}[\overline{\omega}] + \left(\frac{1}{\theta} - \frac{1}{2}\cot\frac{\theta}{2}\right)[\overline{\omega}]^2.$$

Wrenches

- Let $\{a\}$ be a ref. frame and r_a a point in a rigid body. Let the vector $f_a \in \mathbb{R}^3$ represent a force acting on the body. Assume it is a point force acting at r_a.
- The force creates a *torque or moment* $m_a \in \mathbb{R}^3$ given by

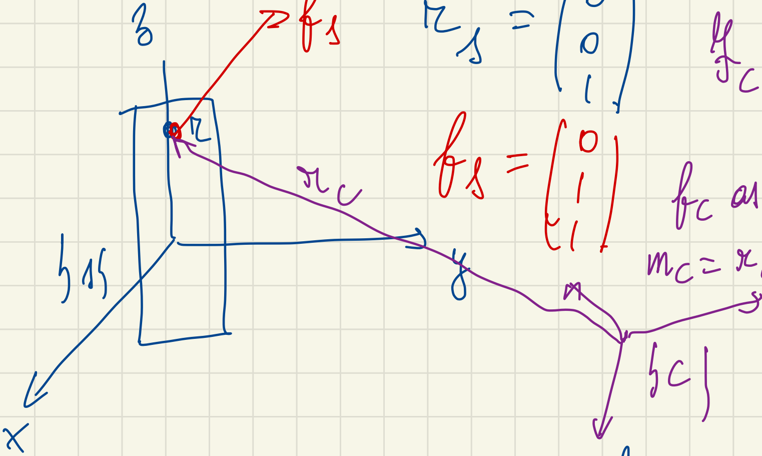
$$\boxed{m_a := r_a \times f_a.}$$



- We merge the force and the moment in a single 6-dimensional vector, called *spatial force* or *wrench*, expressed in $\{a\}$ as

$$\mathcal{F}_a := \begin{bmatrix} m_a \\ f_a \end{bmatrix}. \in \mathbb{R}.$$

- A wrench can contain only a torque and no force: it is called a *pure moment*.
- If there are several forces/moments acting on the body, we can add their corresponding wrenches.



$$r_s = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad f_s = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \qquad \mathcal{F}_c = \begin{pmatrix} m_c \\ f_c \end{pmatrix}$$

f_c as usual

$m_c = r_c \times f_c$

$$\mathcal{F}_s = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \qquad m_s = r_s \times f_s = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

How to obtain $\mathcal{F}$ in a different frame?

recall: $\mathcal{V}_a = Ad_{T_{ab}} \mathcal{V}_b$ pos. of origin of b in a frame

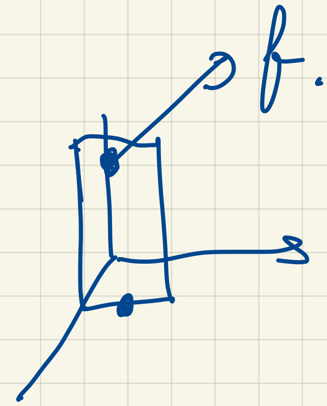
$$\mathcal{V}_a = \begin{pmatrix} \omega_a \\ v_a \end{pmatrix} \qquad \mathcal{V}_b = \begin{pmatrix} \omega_b \\ v_b \end{pmatrix} \qquad T_{ab} = \begin{pmatrix} R_{ab} & p_a \\ 000 & 1 \end{pmatrix}$$

$$Ad_T = \begin{pmatrix} R & 0 \\ [p]R & R \end{pmatrix} \qquad Ad_{T_{ab}}[\mathcal{V}_b] = T_{ab}[\mathcal{V}_b] \underbrace{\,}_{T_{ba}}$$

Power is independent of frame.

"force · speed"

"$\mathcal{F}_s^T \mathcal{V}_s$ = power."



$$= \mathcal{F}_c^T \mathcal{V}_c$$

$$\mathcal{F}_s^T \mathcal{V}_s = \mathcal{F}_c^T \, Ad_{T_{cs}} \mathcal{V}_s$$

or $\mathcal{F}_s^T \, Ad_{T_{sc}} \mathcal{V}_c = \mathcal{F}_c^T \mathcal{V}_c$.

↳ true for all $\mathcal{V}_c$! $\Rightarrow$

$$\mathcal{F}_s^T \, Ad_{T_{sc}} = \mathcal{F}_c^T$$

"duality"

$$\Rightarrow \boxed{\mathcal{F}_c = \left[Ad_{T_{sc}}\right]^T \mathcal{F}_s}$$

$$\left[Ad_{T_{sc}}\right]^T \neq Ad_{T_{cs}} \neq Ad_{T_{sc}^T}$$

Wrenches

- Given 2 frames, a and b, and want to express the wrench in both frames: $\mathcal{F}_a$, $\mathcal{F}_b$.
- Denote by $\mathcal{V}_a$, $\mathcal{V}_b$ the motions induced by the wrench in frames a and b: we know how to relate them: $\mathcal{V}_b = \mathrm{Ad}_{T_{ba}} \mathcal{V}_a$
- Recall: dot product of velocity and force is a power!
- Power is independent from choice of frame!

$$
\begin{aligned}
\mathcal{V}_b^\top \mathcal{F}_b &= \mathcal{V}_a^\top \mathcal{F}_a \\
&= (\mathrm{Ad}_{T_{ab}} \mathcal{V}_b)^\top \mathcal{F}_a \\
&= \mathcal{V}_b^\top (\mathrm{Ad}_{T_{ab}})^\top \mathcal{F}_a
\end{aligned}
$$

- Since the previous equation holds for *all* $\mathcal{V}_b$, we have

$$
\mathcal{F}_b = (\mathrm{Ad}_{T_{ab}})^\top \mathcal{F}_a
$$

$$Ad_{T_{sc}} = \begin{bmatrix} R_{sc} & 0 \\ [p_s] R_{sc} & R_{sc} \end{bmatrix}$$ where $T_{sc} = \begin{bmatrix} R_{sc} & p_s \\ 000 & 1 \end{bmatrix}$

$$\left(Ad_{T_{sc}}\right)^T = \begin{bmatrix} R_{cs} & -R_{cs}[p_s] \\ 0 & R_{cs} \end{bmatrix}$$

$\underbrace{R_{cs}[p_s]^T}_{-[p_s]}$ "

because $R_{sc}^T = R_{cs} = R_{sc}^{-1}$

$[p_s]^T = -[p_s]$

because if $p_s = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$ then $[p_s] =$

$$\begin{bmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{bmatrix}$$

note: recall f_c is f_s in {c} frame

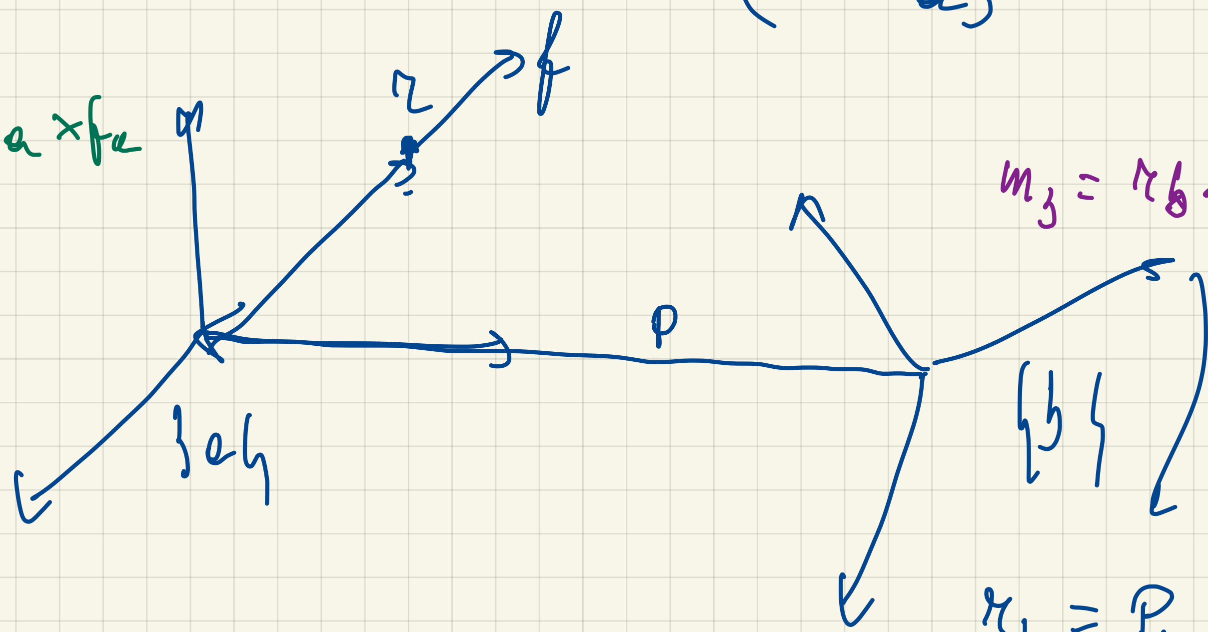
thus $f_c = R_{cs} f_s$.

Introduction to Robotics
Forward Kinematics

MA Belabbas

Exercise: verify $\left(Ad_{T_{ab}}\right)^T \mathcal{F}_a = \mathcal{F}_b$

$m_a = r_a \times f_a$



$m_b = r_b \times f_b$

$r_b = p_b + R_{ba} r_a$

"$[p]\, v = p \times v$"

$[S_1]\ [S_2]\ [S_3]$

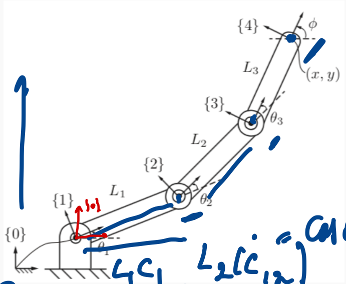
$$\dot{T}_{sb} = [V_s]\, T_{sb}$$

$$\rightsquigarrow T_{sb} = \underbrace{\boxed{e^{[V_s]\theta}}}_{T_{01}} \underbrace{T_{sb}(0)}_{T_1}.$$

Forward kinematics of open chains

$\phi = \theta_1 + \theta_2 + \theta_3$

2D: (x, y, ϕ).



$\theta_3 =$

$L_1 c_1$ $\quad L_2(c_{12})$ $= \cos(\theta_1 + \theta_2)$

$\theta_1 \; \theta_2 \; \theta_3 \rightsquigarrow$ pos. end effector

- The *forward kinematics* of a robot refers to the calculation of the position and orientation of its effector frame from its joint coordinates.
- Here, simple trigonometry yields

$$\rightsquigarrow \left| \begin{aligned} x &= L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) + L_3 \cos(\theta_1 + \theta_2 + \theta_3). \\ y &= L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2) + L_3 \sin(\theta_1 + \theta_2 + \theta_3). \end{aligned} \right.$$

9/30 :- Don't forget to sign up for El w/ CBTF.

∴ - No lecture on Thursday, but Q&A during class time.

Forward kinematics

- For more complex 3D mechanisms, a direct analysis as done in previous slide is too onerous.
- We describe two systematic, principle ways to perform forward kinematics: Product of Exponentials (PoE) and *Denavit-Hartenberg* (next lecture).
- What we can do using previous lectures: attaching frames 1,2,3,4 to the three links and end-effector respectively, and denoting by 0 the reference frame, we need T_{04}, which we can obtain as

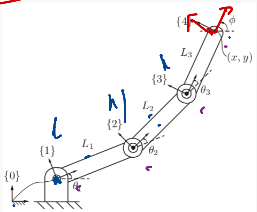
$$\boxed{T_{04} = T_{01}\,T_{12}\,T_{23}\,T_{34},} = \begin{bmatrix} \mathrm{Rot}(\theta_1+\theta_2+\theta_3, \hat{z}) & p \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix}$$

where $p = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$

and $T_{i(i+1)}$ are easy to derive.

$\{0\}$ = space frame $\{1\}$: frame attached to link 1

Forward kinematics: using homogeneous transformations

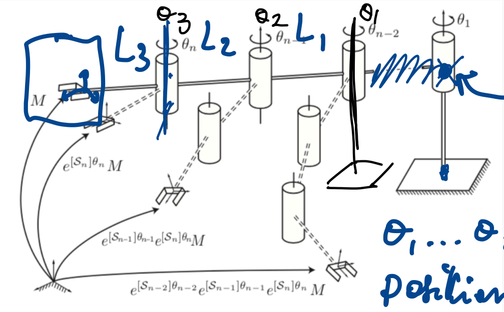


$$T_{01} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, T_{12} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & L_1 \\ \sin\theta_2 & \cos\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{23} = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 & L_2 \\ \sin\theta_3 & \cos\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, T_{34} = \begin{bmatrix} 1 & 0 & 0 & L_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

z axis

$T(\theta_1, \dots \theta_n) \in SE(3)$



$\theta_1 \dots \theta_n = 0$, M is position of end-effector

$$M = \begin{pmatrix} 1 & 0 & 0 & L_1+L_2+L_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Assume the arm is in "resting position": $M := \boxed{T(I, (L_1 + L_2 + L_3, 0, 0)^\top)}$. $\quad T(R, p) := \begin{pmatrix} R & p \\ 000 & 1 \end{pmatrix}$
- From this resting position, what motions are possible?
- Revolute joints allow screw motions with zero pitch.
- Assume $\theta_1 = \theta_2 = 0$. Describe the screw motion of joint 3 by its twist in frame 0: $\mathcal{S}_3 = [\omega_3, v_3]$.

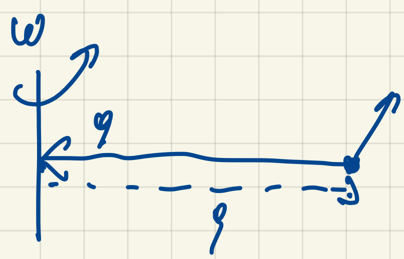
If we actuate (i.e., rotate) joint $n = 3$, how does M move?

joint 3 is a screw motion with parameter $\{q, \hat{s}, h\}$

$$q = \begin{pmatrix} L_1 + L_2 \\ 0 \\ 0 \end{pmatrix}$$

$$\hat{s} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad h = 0.$$

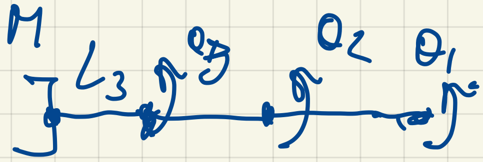
Twist: $\hat{\omega} = \hat{s}$; $v_s = \begin{pmatrix} L_1 + L_2 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$



$$v_3 = \omega \times (-q). = \begin{pmatrix} 0 \\ -[L_1 + L_2] \\ 0 \end{pmatrix}.$$

$$[S_3] = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -(L_1 + L_2) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow e^{[S_3]\theta_3}$$

if joint 3 is actuated, $M \rightarrow e^{[S_3]\theta_3} M.$



$$M = \begin{pmatrix} 1 & & & L_1 + L_2 + L_3 \\ & 1 & & 0 \\ & & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

joint 3 is a rotation around z axis

$$\mathrm{Rot}(\hat{z}, \theta_3) = \begin{pmatrix} c_{\theta_3} & -s_{\theta_3} & 0 & 0 \\ s_{\theta_3} & c_{\theta_3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M \to \underbrace{\cancel{\mathrm{Rot}(\hat{z}, \theta_3)}}_{e^{[S_3]\theta_3}} M.$$

$$\underbrace{e^{[S_3]\theta_3}}_{\hookrightarrow \text{Rodrigues}} M \qquad \begin{bmatrix} \mathrm{Rot}(\hat{z}, \theta_3) & \vdots \\ 0 \; 0 \; 0 & 1 \end{bmatrix}$$

---

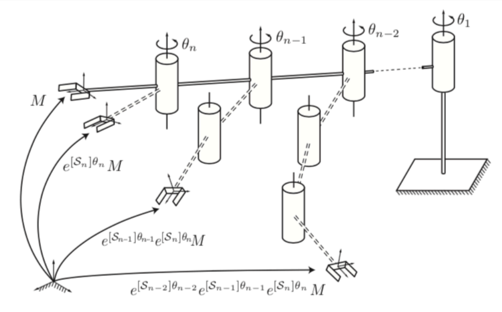
joint 2 is activated, it acts on joints 3 & M.

$$e^{[S_3]\theta_3} M \to e^{[S_2]\theta_2} e^{[S_3]\theta_3} M.$$

where $S_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -L_1 \\ 0 \end{bmatrix}$, $S_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$$\Rightarrow T(\theta_1, \theta_2, \theta_3) = e^{[S_1]\theta_1} e^{[S_2]\theta_2} e^{[S_3]\theta_3} M.$$

Forward kinematics: using screw motions



- How to obtain ω_3, v_3?
- Revolute joint: rotation around axis perpendicular to motion and ω is normalized vector in this direction $\longrightarrow \omega_3 = (0, 0, 1)^\top$.
- To find v_3, consider a rigid body attached to the joint and following the joint's motion: v_3 is the velocity of the point at the origin of frame 0. This speed is $\omega \times (-L_1 - L_2, 0, 0)^\top$.

Forward kinematics: using screw motions

- When only θ_3 is allowed to move, we have

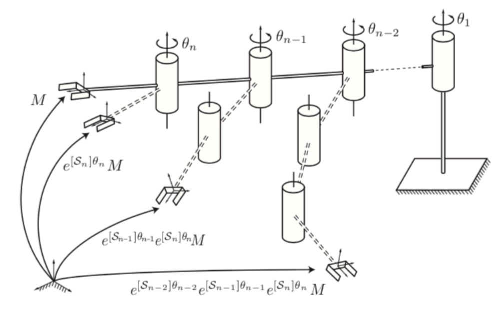
$$T_{04} = e^{[\mathcal{S}_3]\theta_3} M,$$

where, by definition,

$$[\mathcal{S}_3] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 & v_1 \\ \omega_3 & 0 & -\omega_1 & v_2 \\ -\omega_2 & \omega_1 & 0 & v_3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

- Thus here, $[\mathcal{S}_3] = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -(L_1 + L_2) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

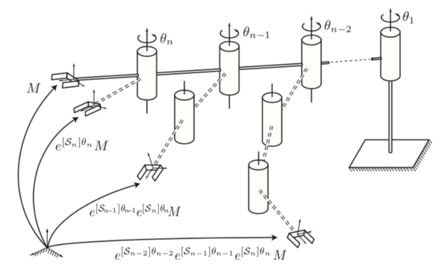
Forward kinematics: product of exponentials



- We now repeat the procedure: assume θ_1 fixed at zero, θ_3 fixed at an arbitrary value, and move θ_2. The corresponding twist is

$$\mathcal{S}_2 = (\omega_2 = (0, 0, 1)^\top, (0, -L_1, 0))^\top.$$

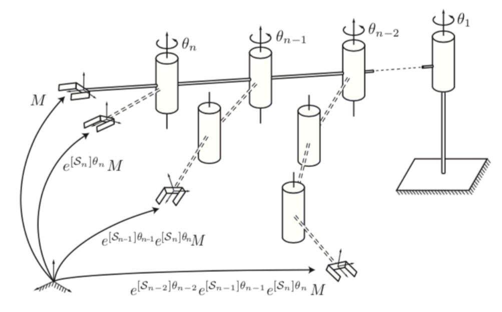
Forward kinematics: product of exponentials



- Thus $[\mathcal{S}_2] = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -L_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
- Now rotation about joint 2 can be viewed as applying a screw motion to the rigid body (link 1+ link 2), thus

$$T_{04} = e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M.$$

Forward kinematics: product of exponentials



- Finally, $\mathcal{S}_1 = ((0,0,1)^\top, (0,0,0)^\top)$ and

$$T_{04} = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M.$$

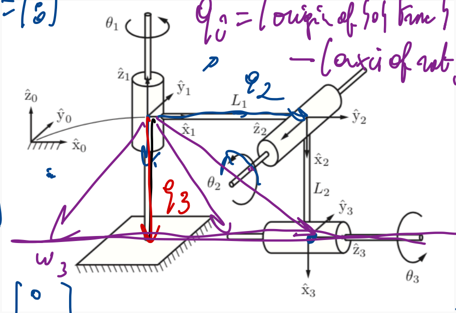
Product of exponentials: example

3R $\quad \omega_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad q_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$S_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad S_2 : \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ -L_1 \end{bmatrix}$

q_i = (origin of {0} frame) − (axis of rot)

$M = \begin{bmatrix} 0 & 0 & 1 & L_1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$



$v_2 = -\omega_2 \times q_2$

$= \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \times \begin{pmatrix} L_1 \\ 0 \\ 0 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \\ -L_1 \end{bmatrix}$

S_3 :

$\omega_3 \quad q_3$

V_3

$\omega_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad q_3 = \begin{pmatrix} 0 \\ 0 \\ -L_2 \end{pmatrix}$

- 3R open chain, with non-collinear rotation axes.
- Note: $\hat{z}_i$ aligned with rotation axis, $\hat{x}_i$ points to next joint.
- Forward kinematics as the form

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M.$$

We need to find M and the $\mathcal{S}_i$'s.

$v_3 = \begin{bmatrix} 0 \\ -L_2 \\ 0 \end{bmatrix}$

$= -\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ -L_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -L_2 \\ 0 \end{pmatrix}$ $\checkmark$

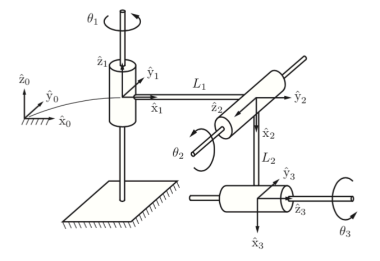
recall $v_i = -w_i \times q_i$

$$\Rightarrow v_i = -w_i \times (q_i + \alpha w_i)$$

$$= -w_i \times q_i$$

$$-\alpha\, w_i \cancelto{0}{\times w_i}$$

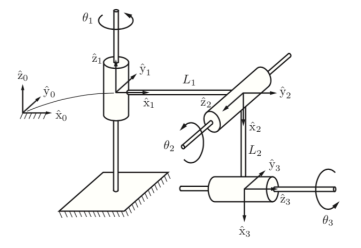
Product of exponentials: example



- M is the configuration of the end-effector fixed frame (frame 3) when *all* joint variables are zero.
- We obtain

$$M = \begin{bmatrix} 0 & 0 & 1 & L_1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Product of exponentials: example



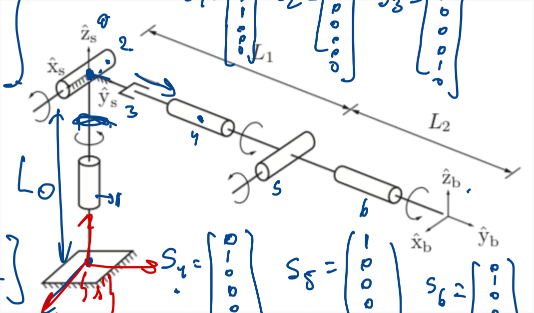
- The screw axis for joint 1 in frame 0 is $\mathcal{S}_1 = (\omega_1, v_1)$ with $\omega_1 = (0, 0, 1)^\top$ and $v_1 = (0, 0, 0)^\top$.
- The screw axis for joint 2 in frame 0 is $\mathcal{S}_2 = (\omega_2, v_2)$ with $\omega_2 = (0, -1, 0)^\top$ and $v_2 = (0, 0, -L_1)^\top$.
- To obtain v_2, set q to be the vector joining origin of reference frame to center of joint, here $q = (L_1, 0, 0)^\top$ and then $v_2 = \omega_2 \times (-q)$.
- Finally, $\mathcal{S}_3 = (\omega_3, v_3)$ with $\omega_3 = (1, 0, 0)^\top$ and $v_3 = \omega_3 \times (-(L_1, 0, -L_2)^\top) = (0, -L_2, 0)$

Product of exponentials: example: RRPRRR open chain

$$[S] = \begin{bmatrix} [\omega] & v \\ 0\,0\,0 & 0 \end{bmatrix} \qquad [\omega] = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

$$S_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad S_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad S_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad S_i = \begin{pmatrix} \omega_i \\ v_i \end{pmatrix}$$

$$T = e^{[S_1]\theta_1} \cdots e^{[S_6]\theta_6} M$$



$$v_4 = -\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ L_1/2 \\ 0 \end{bmatrix} \quad S_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad S_5 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -L_1 \end{pmatrix} \quad S_6 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$M = \begin{bmatrix} I & \begin{matrix} 0 \\ L_1+L_2 \\ 0 \end{matrix} \\ 0\,0\,0 & 1 \end{bmatrix} \qquad e^{[S]} = \begin{bmatrix} e^{[\omega]\theta} & G(\theta)v \\ 0\,0\,0 & 1 \end{bmatrix}$$

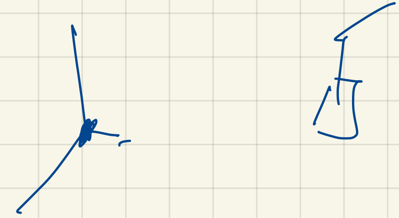
Forward kinematics is described by

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & L_1 + L_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

i	ω_i	v_i
1	$(0,0,1)$	$(0,0,0)$
2	$(1,0,0)$	$(0,0,0)$
3	$(0,0,0)$	$(0,1,0)$
4	$(0,1,0)$	$(0,0,0)$
5	$(1,0,0)$	$(0,0,-L_1)$
6	$(0,1,0)$	$(0,0,0)$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

We have computed

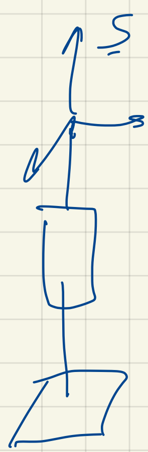


$$T_{sb} = e^{[S_1]\theta_1} \cdots e^{[S_6]\theta_6} M.$$

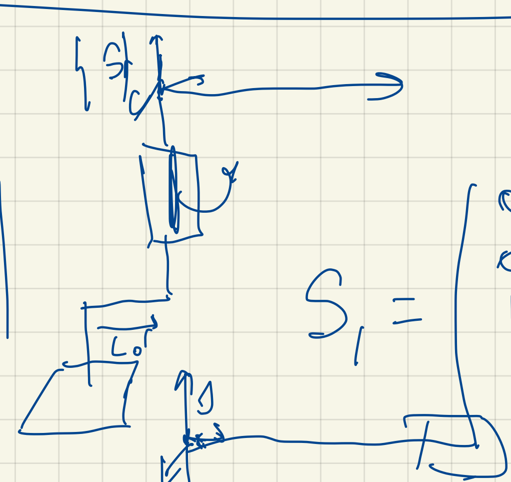
I want $T_{s'b}$

$$\rightsquigarrow T_{s'b} = T_{s's}\, T_{sb}$$

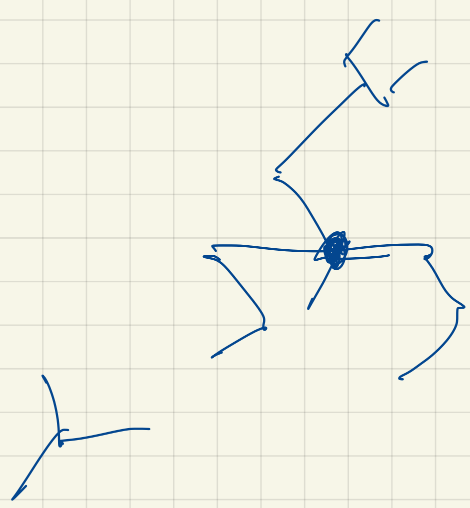
$$T_{s's} = \begin{bmatrix} I & \begin{matrix} 0 \\ 0 \\ L_0 \end{matrix} \\ 0\ 0\ 0 & 1 \end{bmatrix}$$

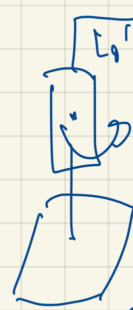


L_0



$$S_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$





$$S_i = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -L_0' \\ 0 \end{pmatrix}$$

$\theta_i(t)$

$$\dot{e}^{[S_i]\theta_i} = \dot{\theta}_i [S_i] e^{[S_i]\theta_i}$$