$$\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \rightsquigarrow [\omega] := \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

$$\to \boxed{\frac{d}{dt} R = [\omega] R}$$

Rotation

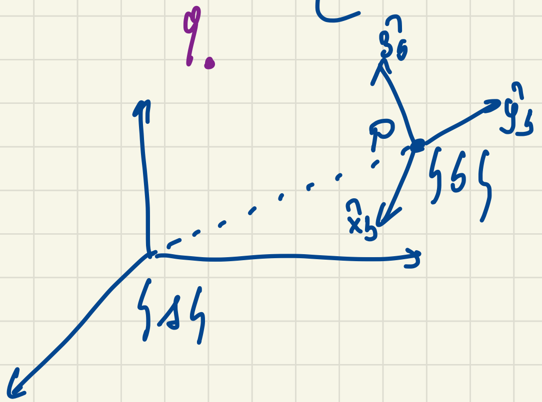
$$\boxed{\frac{d}{dt} x(t) = v}$$

Cartesian.

THU 9/11

- HW 2 is on PL (due Fri 9/19)

$$T_{sb} = \begin{bmatrix} R_{sb} & p_s \\ 0\,0\,0 & 1 \end{bmatrix}$$



p_s = coord of p in $\{s\}$ frame

$$R_{sb} = \begin{bmatrix} \hat{x}_b & \hat{y}_b & \hat{z}_b \end{bmatrix}$$

q has coord q_s in $\{s\}$ frame.

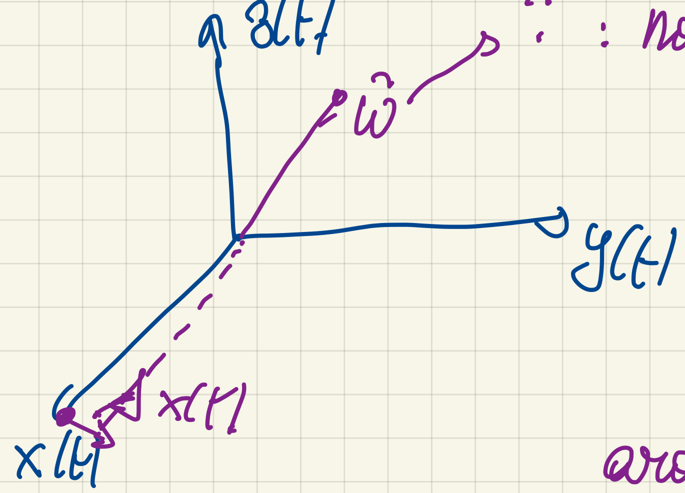
q_b in $\{b\}$ frame.

→ $\boxed{q_s = T_{sb}\, q_b}$

$$\underbrace{T_{sb}^{-1}}_{\cancel{\text{inv}}} = T_{bs} = \begin{bmatrix} \overset{R_{sb}^T}{\overset{\parallel}{R_{bs}}} & -R_{bs}\, p_s \\ 0\,0\,0 & 1 \end{bmatrix}$$

⇝ explicit!

$R_{sb}(t)$



$\hat{\ }$: normalized (i.e $\|\hat{\omega}\|=1$)

after Δt, for a rotation around $\hat{\omega}$,

$\Delta x(t) = \hat{\omega} \times x(t)$ $\quad x(t) \rightsquigarrow x(t+\Delta t) = x(t) + \hat{\omega} \times x(t)$

$$\dot{R}_{sb} = \frac{d}{dt} R_{sb} = \begin{bmatrix} \dot{x}_b(t) & \dot{y}_b(t) & \dot{z}_b \end{bmatrix}$$

$$= \begin{bmatrix} \omega \times x_b & \omega \times y_b & \omega \times z_b \end{bmatrix}$$

$$= \underline{\omega \times} R_{sb}$$

We want to use matrix product and not cross-product.

$$\omega = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \text{ then } [\omega] := \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

then, $\boxed{\omega \times p = [\omega] p}$ for any ω & p

$$\therefore \quad \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \times \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{bmatrix} \omega_2 p_3 - p_2 \omega_3 \\ \omega_3 p_1 - \omega_1 p_3 \\ \omega_1 p_2 - \omega_2 p_1 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} \omega_2 p_3 - p_2 \omega_3 \\ \vdots \\ \vdots \end{bmatrix}$$

$$\rightsquigarrow \quad \frac{d}{dt} R_{sb}(t) = [\omega_s] R_{sb} \qquad (*)$$

Solution of $(*)$ is

$$\boxed{R_{sb}(t) = e^{[\omega_s]t} R_{sb}(0)}$$

$\underline{e^{[\omega_s]t\theta}}$ is a MATRIX EXPONENTIAL

in general, for $A \in \mathbb{R}^{n \times n}$

$$\left[e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots \right]$$

in general if $\underbrace{\dot{R} = AR}_{R(0)=R_0.}$, then $R(t) = \underline{e^{At}} R(0)$

<u>check</u>: if $R(t) = e^{At} R(0)$, then

$$\dot{R}(t) = \frac{d}{dt}\left\{ \left[I + At + \frac{A^2t^2}{2!} + \cdots \right] R(0) \right\}$$

$$= \left[0 + A + \frac{2A^2t}{2!} + \frac{3A^3t^2}{3!} + \cdots \right] R(0)$$

$$= A \underbrace{\left[I + At + \frac{A^2t^2}{2!} + \cdots \right] R(0)}_{R(t)}$$

- $e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots$

- Properties:
- $\frac{d}{dt}\left(e^{At}\right) = \boxed{Ae^{At} = e^{At}A.}$

$$\frac{d}{dt}\left(I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!}\right)$$

$$= \left(A + A^2t + \frac{A^3t^2}{2!}\right)$$

$$= A\left(I + At + \frac{A^2t^2}{2!} + \dots\right) = \left(I + At + \dots\right)A.$$

- We say that A and e^{At} commute!

⚠ in general, for $A, B \in \mathbb{R}^{n \times n}$

$$AB \neq BA$$

- $e^{A+B} = e^{A}e^{B} = e^{B}e^{A}$
  if and only if A & B commute
  ie $AB = BA$.

- if P is st. $PDP^{-1} = A$
  for D a diagonal matrix.
  [We say P diagonalizes A]
  then $\boxed{e^{At} = Pe^{Dt}P^{-1}}$ ✓

$$e^{Dt} = I + D + \frac{D^2t^2}{2!} + \cdots$$

but if $D = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix}$, then $D^{\cancel{2}\,k} = \begin{bmatrix} d_1^{\cancel{2}\,k} & & \\ & \ddots & \\ & & d_n^{\cancel{2}\,k} \end{bmatrix}$

$$\Rightarrow e^{Dt} = \begin{bmatrix} e^{d_1 t} & & \\ & \ddots & \\ & & e^{d_n t} \end{bmatrix}$$

$$(e^{At})^{-1} = e^{-At} \qquad e^{At} e^{-At} = e^{At-At} = e^{0}$$

$$A(-A) \stackrel{?}{=} (-A)A \quad \checkmark$$

(implies that $e^{0} = I$)

matrix exponential is <u>always</u> invertible

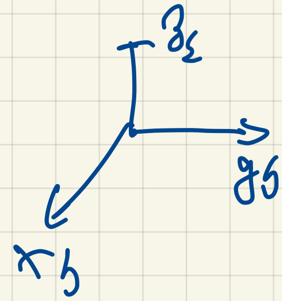
- $$\frac{d}{dt} R_{sb}(t) = [\omega] R_{sb}(t)$$

$$R_{sb}(t) = e^{[\omega]t} R_{sb}(0)$$

$\therefore$ $\|\omega\| = 1$

(ω, t) or (ω, θ) are the exponential coordinates of a rotation.

recall: $R_{sb} = \begin{bmatrix} x_b & y_b & z_b \end{bmatrix}$



$$R_{sb}^T R_{sb} = I \qquad \det R_{sb} = 1$$

$$R_{sb} = \begin{bmatrix} \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{bmatrix}$$

$$R_{se} = e^{[\omega]_\times t}$$

$$(\omega, t) \rightsquigarrow \overbrace{e^{[\omega]t}}^{R_{sb}} R_0$$

chosen freely

Angular Velocities in Reference Frame

$\frac{d}{dt} R_{sb} = [\omega_s] R_{sb}$ $\rightsquigarrow$ $R_{sb} = e^{[\omega]t} R_{sb}(0)$ $\rightsquigarrow I$

$R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ \vdots & \ddots & \\ \vdots & & r_{31} \end{pmatrix}$

Some useful properties and relations

Given any $\omega \in \mathbb{R}^3$ and $R \in SO(3)$, the following holds: $\boxed{R[\omega]R^\top = [R\omega]}$ (1)

proof: just plug in values

Now recall: $[\omega_s] = \dot{R}R^\top$

$\dot{R}_{sb} = [\omega_s] R_{sb} \Rightarrow [\omega_s] = \dot{R}_{sb} R_{sb}^\top = \dot{R}_{sb} R_{bs} = \dot{R}_{ss}$

$\boxed{\omega_s = R_{sb}\omega_b}$

If R is R_{sb}, we have that $\omega_s = R\omega_b \Leftrightarrow \omega_b = R^\top \omega_s$

$$[\omega_b] = [R^\top \omega_s] = R^\top(\dot{R}R^\top)R = R^\top \dot{R} = R^{-1}\dot{R}$$

use (1) with $R \leftarrow R^\top$

$[\omega_b] = R_{bs} \dot{R}_{sb}$

This gives us: $[\omega_s] = \dot{R}R^\top$ and $[\omega_b] = R^\top \dot{R}$

$[\omega_s] = \dot{R}_{sb} R_{bs}$ $\quad [\omega_b] = R_{bs} \dot{R}_{sb}$

Introducing
Exponential Coordinates

Recall the Matrix Exponential

If $\dot{x}(t) = Ax(t), x(0) = x_0$, where $x \in \mathbb{R}^n, A \in \mathbb{R}^{n\times n}$, the solution to the ODE is:

$$x(t) = e^{At}x_0$$

The matrix exponential / the exponential function can be expanded:

$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots$$

Recall the Matrix Exponential

If $\dot{x}(t) = Ax(t), x(0) = x_0$, where $x \in \mathbb{R}^n, A \in \mathbb{R}^{n\times n}$, the solution to the ODE is:

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The matrix exponential / the exponential function can be expanded:

$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots$$

Properties:

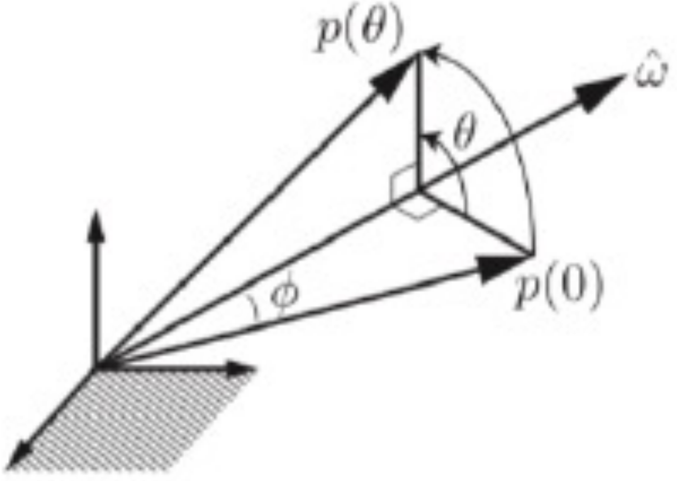
1. $\frac{d}{dt}(e^{At}) = Ae^{At} = e^{At}A$
2. If $A = PDP^{-1}$, then $e^{At} = Pe^{Dt}P^{-1}$
3. If $AB = BA$, then $e^A e^B = e^B e^A = e^{A+B}$
4. Invertible: $(e^{At})^{-1} = e^{-At}$

Exponential Coordinate Representation

- Instead representing orientation as a rotation matrix, we introduce a three-parameter representation: **exponential coordinates**
- Recall: $\widehat{\omega}$ rotation axis, θ angle of rotation
- Then $\widehat{\omega}\theta \in \mathbb{R}^3$gives the exponential coordinate representation

- A new interpretation for a frame coincident with $\{s\}$:
  - A rotation for 1 second around $\widehat{\omega}$ at angular velocity θ, then the resulting frame is R
  - A rotation for θ seconds around $\widehat{\omega}$ at angular velocity 1, then the resulting frame is R

Exponential Coordinates of Rotations



* RODRIGUES FORMULA

$$e^{[\omega]t} = I + [\omega]t + \frac{[\omega]^2 t^2}{2!} \cdots + \cdots$$

$\rightarrow \omega_1^2 + \omega_2^2 + \omega_3^2 = 1$

for $[\omega]$ with $\|\omega\| = 1$, $\quad e^{[\omega]t} = I + \sin t\,[\omega] + (1 - \cos t)[\omega]^2$

$\theta \qquad \theta \qquad \theta$

. if $\|\omega\| = 1$, then

$$\overset{[\omega]}{\begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}} \overset{[\omega]}{\begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}} = \begin{bmatrix} 1 - \omega_1^2 & \omega_1\omega_1 & \cdots \\ & & \ddots \\ & & \end{bmatrix} = [\omega]^2$$

$$[\omega]^3 = [\omega]^2[\omega] = -[\omega] \; ; \; [\omega]^4 = -[\omega]^2$$

$$e^{[\omega]\theta} = I + [\omega]\theta + [\omega]^2\frac{\theta^2}{2!} + [\omega]^3\frac{\theta^3}{3!} + [\omega]^4\frac{\theta^4}{4!} + \dots$$

$$[\omega]^3 = -[\omega] \qquad [\omega]^4 = -[\omega]^2$$

$$\Rightarrow e^{[\omega]\theta} = I + [\omega]\theta + [\omega]^2\frac{\theta^2}{2!}$$

$$- \frac{\theta^3}{3!} - \frac{\theta^4}{4!}$$

$$+ \frac{\theta^5}{5!} + \frac{\theta^6}{6!}$$

$$= I + [\omega] \sin\theta + [\omega]^2 (1-\cos\theta)$$

$$e^{[\omega]\theta} = I + \sin\theta\, [\omega] + (1-\cos\theta)[\omega]^2$$

$\omega, \theta \rightsquigarrow e^{[\omega]\theta}$

$= I + \sin\theta\,[\omega] + [1-\cos\theta][\omega]^2$

MATRIX EXPONENTIAL.

- MATRIX LOGARITHM.

$\underline{R} \rightsquigarrow (\underline{\omega}, \underline{\theta})$ st $e^{[\omega]\theta} = R$.

[matrix exp. expm

$\rightsquigarrow$ logm

Matrix Logarithm

- Given rotation matrix R, we need to take the *logarithm* to find the exponential coordinates:

$$\text{exp:} \quad [\hat{\omega}]\theta \in so(3) \quad \rightarrow R \in SO(3)$$

$$\text{log:} \quad R \in SO(3) \quad \rightarrow [\hat{\omega}]\theta \in so(3)$$

Matrix Logarithm

$\operatorname{tr} R = 2c_\theta + 1$

$= r_{11} + r_{22} + r_{33}$

- Given rotation matrix R, we need to take the *logarithm* to find the exponential coordinates:

$e^{[\hat\omega]\theta}$

exp:	$[\widehat{\omega}]\theta \in so(3)$	$\rightarrow R \in SO(3)$
log:	$R \in SO(3)$	$\rightarrow [\widehat{\omega}]\theta \in so(3)$

$$\begin{cases} r_{11} = c_\theta + \hat\omega_1^2(1-c_\theta) \\ r_{12} = \hat\omega_1\hat\omega_2(1-c_\theta) - \hat\omega_3 s_\theta \\ \vdots \end{cases}$$

- If we expand Rodrigues formula:

$$\underline{\operatorname{Rot}(\widehat{\omega},\theta) = I + \sin\theta\,[\widehat{\omega}] + (1-\cos\theta)[\widehat{\omega}]^2}$$

$$\operatorname{Rot}(\widehat{\omega},\theta) = \begin{bmatrix} c_\theta + \widehat{\omega}_1^2(1-c_\theta) & \widehat{\omega}_1\widehat{\omega}_2(1-c_\theta) - \widehat{\omega}_3 s_\theta & \widehat{\omega}_1\widehat{\omega}_3(1-c_\theta) + \widehat{\omega}_2 s_\theta \\ \widehat{\omega}_1\widehat{\omega}_2(1-c_\theta) + \widehat{\omega}_3 s_\theta & c_\theta + \widehat{\omega}_2^2(1-c_\theta) & \widehat{\omega}_2\widehat{\omega}_3(1-c_\theta) - \widehat{\omega}_1 s_\theta \\ \widehat{\omega}_1\widehat{\omega}_3(1-c_\theta) - \widehat{\omega}_2 s_\theta & \widehat{\omega}_2\widehat{\omega}_3(1-c_\theta) + \widehat{\omega}_1 s_\theta & c_\theta + \widehat{\omega}_3^2(1-c_\theta) \end{bmatrix}$$

$$= R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$$

Matrix Logarithm Method

- If we take the trace of the matrix, we can solve for θ:
  - $tr(R) \coloneqq r_{11} + r_{22} + r_{33} = 1 + 2\cos\theta$

$\Rightarrow \quad \theta = \cos^{-1}\left[\frac{\operatorname{tr} R - 1}{2}\right]$

$$\mathrm{Rot}(\hat{\omega}, \theta) = \begin{bmatrix} c_\theta + \hat{\omega}_1^2(1-c_\theta) & \hat{\omega}_1\hat{\omega}_2(1-c_\theta) - \hat{\omega}_3 s_\theta & \hat{\omega}_1\hat{\omega}_3(1-c_\theta) + \hat{\omega}_2 s_\theta \\ \hat{\omega}_1\hat{\omega}_2(1-c_\theta) + \hat{\omega}_3 s_\theta & c_\theta + \hat{\omega}_2^2(1-c_\theta) & \hat{\omega}_2\hat{\omega}_3(1-c_\theta) - \hat{\omega}_1 s_\theta \\ \hat{\omega}_1\hat{\omega}_3(1-c_\theta) - \hat{\omega}_2 s_\theta & \hat{\omega}_2\hat{\omega}_3(1-c_\theta) + \hat{\omega}_1 s_\theta & c_\theta + \hat{\omega}_3^2(1-c_\theta) \end{bmatrix}$$

r_{12}

r_{21}

$\rightarrow \quad \underline{r_{21} - r_{12}} = \underline{\hat{\omega}_3} \cdot 2 s_\theta$

Matrix Logarithm Method

- If we take the trace of the matrix, we can solve for θ:
  - $tr(R) \coloneqq r_{11} + r_{22} + r_{33} = 1 + 2\cos\theta$
- If we compute $R - R^\top$, we get:
  - $r_{32} - r_{23} = 2\hat{\omega}_1 \sin\theta$
  - $r_{13} - r_{31} = 2\hat{\omega}_2 \sin\theta$
  - $r_{21} - r_{12} = 2\hat{\omega}_3 \sin\theta$

$$\left.\begin{aligned} r_{32} - r_{23} &= 2\hat{\omega}_1 \sin\theta \\ r_{13} - r_{31} &= 2\hat{\omega}_2 \sin\theta \\ r_{21} - r_{12} &= 2\hat{\omega}_3 \sin\theta \end{aligned}\right\} \quad [\hat{\omega}] = \frac{1}{2\sin\theta}(R - R^\top)$$

Matrix Logarithm Method

- If we take the trace of the matrix, we can solve for θ:
  - $tr(R) \coloneqq r_{11} + r_{22} + r_{33} = 1 + 2\cos\theta$
- If we compute $R^\top - R$, we get:
  - $r_{32} - r_{23} = 2\widehat{\omega}_1 \sin\theta$
  - $r_{13} - r_{31} = 2\widehat{\omega}_2 \sin\theta$
  - $r_{21} - r_{12} = 2\widehat{\omega}_3 \sin\theta$

$$\left.\begin{aligned} r_{32} - r_{23} &= 2\widehat{\omega}_1 \sin\theta \\ r_{13} - r_{31} &= 2\widehat{\omega}_2 \sin\theta \\ r_{21} - r_{12} &= 2\widehat{\omega}_3 \sin\theta \end{aligned}\right\} \quad [\widehat{\omega}] = \frac{1}{2\sin\theta}(R - R^\top)$$

- But what if $\sin\theta = 0$? (called singularities)
  - If $\theta = 2k\pi$, we have rotated by 360 degrees
  - If $\theta = (2k+1)\pi$, then Rodrigues formula is $R = I + 2[\widehat{\omega}]^2$, which gives:

$\boxed{\widehat{\omega}_i = \pm\sqrt{\frac{r_{ii}+1}{2}}}$ and $2\widehat{\omega}_i\widehat{\omega}_j = r_{ij}$ for $i \neq j$

$\theta = k\pi$

$\sin\theta = 0$

$$\begin{bmatrix} 0 & -\omega_3 & \omega_1 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_1 & \omega_1 & 0 \end{bmatrix}^2 = \dots$$

$$^3 = -[\omega]$$

$$\boxed{\omega_1^2 + \omega_2^2 + \omega_3^2 = 1}$$

$$(\omega, \theta) \rightsquigarrow R = e^{[\omega]\theta}$$

$$\underset{\sim}{R} \rightsquigarrow \underline{\omega, \theta}$$

$$(x \; y \; z)$$

$$[\omega]\theta$$

$$[\omega] = \left[\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}\right] = \begin{bmatrix} 0 & -\omega_3 & \omega_1 \\ & 0 & -\omega_2 \\ & & 0 \end{bmatrix}$$

$$A e^{At} = e^{At} A$$

$$\overset{\cancel{A} \; BA}{AB}$$

$$A\left(I + \overset{B}{\cancel{A}}t + \frac{\overset{B^2}{\cancel{A}^2} t^2}{2!} + \dots\right) = \left(A + A^2 t + \frac{A^3 t^2}{2!} + \dots\right)$$

$$= (I + At + \dots)\, A$$

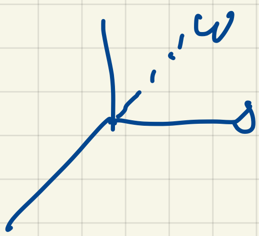
On to Homogeneous Transformations!

Exponential coord. for rotations

- $\frac{d}{dt} R_{sb}(t) = [\omega] R_{sb}(t)$

  where $[\omega] = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$

  for $\omega = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$



$\rightsquigarrow$ solution is $R_{sb}(t) = e^{[\omega]t} R(0)$

exp coord. $(\hat{\omega}, \theta)$ where

$\|\hat{\omega}\| = 1$, θ is the angle of rot.

↳ axis of rotation.

- $e^{[\omega]t} := I + [\omega]t + [\omega]^2 \frac{t^2}{2!} + [\omega]^3 \frac{t^3}{3!} + \dots$

- prop of matrix exp.

$$e^{A+B} = e^A e^B = e^B e^A$$

IFF $\underbrace{AB = BA}_{\text{if so, we say that } A \& B \text{ commute.}}$

if $D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$, then $e^D = \begin{pmatrix} e^{d_1} & & \\ & \ddots & \\ & & e^{d_n} \end{pmatrix}$

pf: $e^D = I + D + \frac{D^2}{2!} + \frac{D^3}{3!} + \dots$

$$= \begin{pmatrix} \underbrace{1 + d_1 + \frac{d_1^2}{2!}}_{e^{d_1}} & & \\ & \ddots & \\ & & \underbrace{1 + d_n + \frac{d_n^2}{2!} + \dots}_{e^{d_n}} \end{pmatrix}$$

$D^2 = \begin{pmatrix} d_1^2 & & \\ & \ddots & \\ & & d_n^2 \end{pmatrix}$, $D^k = \begin{pmatrix} d_1^k & & \\ & \ddots & \\ & & d_n^k \end{pmatrix}$

- $A = PDP^{-1}$, then

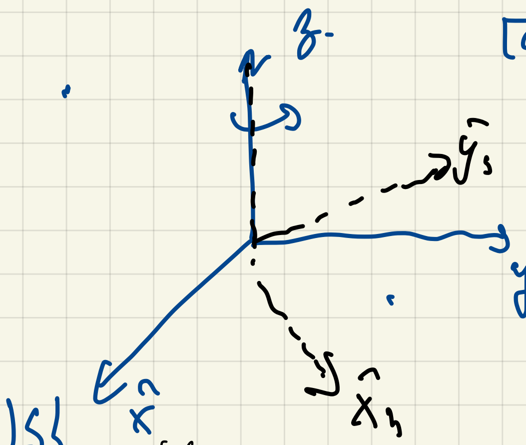
$$e^A = P e^D P^{-1}$$

Rodrigues' formula.

only true for $[\hat{\omega}]$ normalized

$$\boxed{e^{[\hat{\omega}]\theta} = I + \sin\theta [\hat{\omega}] + (1 - \cos\theta)[\hat{\omega}]^2}$$

example:



rotate around $\hat{z}$ axis for $45°$

$$\hat{\omega} = \begin{array}{c} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{array} \begin{array}{l} \to \omega_1 \\ \to \omega_2 \\ \to \omega_3 \end{array} \qquad [\hat{\omega}] = \begin{array}{cc} \begin{array}{ccc} & -\omega_3 & \omega_2 \end{array} & \\ \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \to -\omega_1 \end{array}$$

$$I + \sin\theta [\hat{\omega}] + (1-\cos\theta)[\hat{\omega}]^2$$

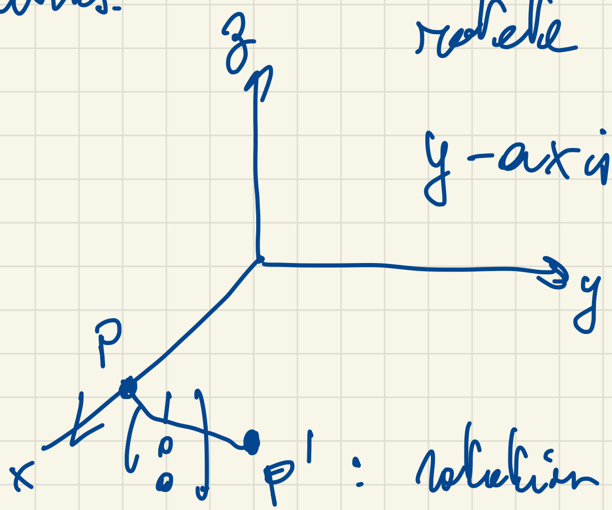
$$[\hat{\omega}]^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$[\hat{\omega}]$

$$\to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{\sqrt{2}}{2}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \left(1-\frac{\sqrt{2}}{2}\right)\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{2}}{2} & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = R_{sb}.$$

- Rotation matrices are reference frames.
- Use them to change frames: $R_{sb} R_{bc} = R_{sc}$
- They can be used to "rotate" points.



rotate $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ around y-axis by 45°.

p' : rotation of p around y by 45°.

$\therefore \; e^{[\hat{\omega}]\theta} p$ with $\hat{\omega} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \theta = 45$

$$[\hat{\omega}] = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad [\hat{\omega}]^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$e^{[\hat{\omega}]\theta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \frac{\sqrt{2}}{2}$$

$$+ \left(1 - \frac{\sqrt{2}}{2}\right) \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \sqrt{2}/2 \\ 0 & 1 & 0 \\ -\sqrt{2}/2 & 0 & \sqrt{2}/2 \end{bmatrix}$$

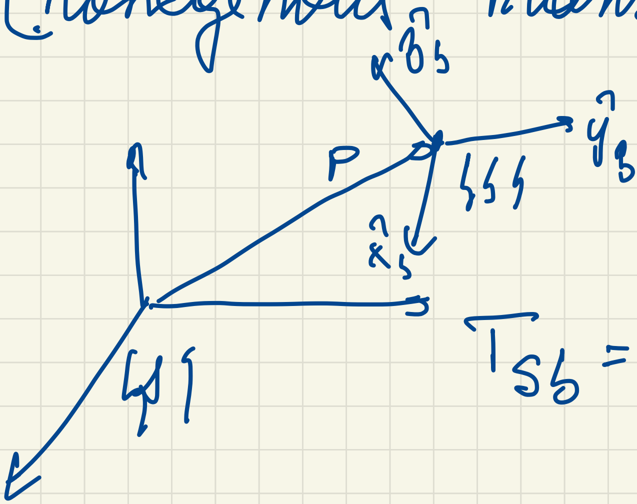
thus, $p' = e^{[\hat{\omega}]\theta} \underbrace{p}_{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}} = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ -\frac{\sqrt{2}}{2} \end{pmatrix}$

- If R_{sb} is obtained by rotating $\underset{\text{{sb}-frame}}{I}$ around $\hat{\omega}$ by θ degree, $R_{sb} = e^{[\hat{\omega}]\theta}$
- If we rotate a point with coord p by

an angle θ around axis $\hat{\omega}$,

then p goes to $p' = e^{[\hat{\omega}]\theta} p$

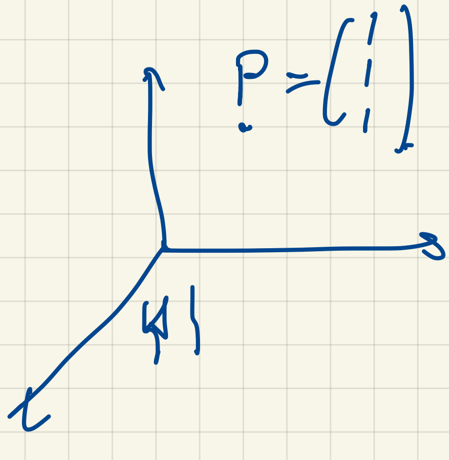
Exponential coord for homogeneous transformations.



$$T_{sb} = \begin{pmatrix} \hat{x}_b & \hat{y}_b & \hat{z}_b & p_b \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T_{sb} T_{bc} = T_{sc}.$$

· We can use homogeneous transformation matrices to rotate & translate points.



I want to translate $p \rightsquigarrow p'$ distance of 2 in direction v.

$$p' = p + 2v.$$

$$T = \begin{pmatrix} 1 & 0 & 0 & \\ 0 & 1 & 0 & 2v \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

for implements a rotation by θ degrees & translation by $2v$.

$$\underbrace{\begin{pmatrix} I & 2v \\ 000 & 1 \end{pmatrix}}_{T} \begin{pmatrix} p \\ 1 \end{pmatrix} = \begin{pmatrix} p + 2v \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} p' \\ 1 \end{pmatrix}}$$

$\begin{pmatrix} p \\ 1 \end{pmatrix}$ is the homogeneous coord of p.

$I =$ rotation by θ degree

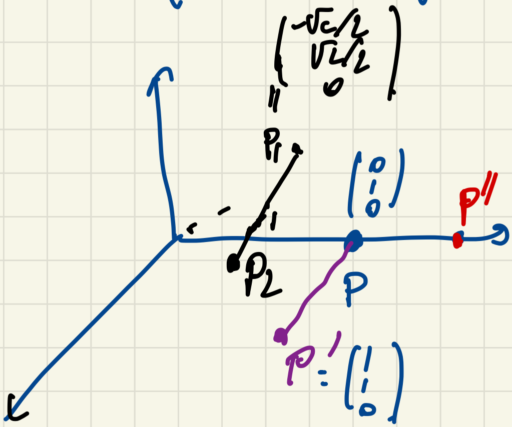
$$= e^{[\hat{\omega}]\theta} = I + \sin\theta\,[\omega] + (1 - \cos\theta)[\omega]^2$$

$$T = \begin{pmatrix} R & v \\ 0\,0\,0 & 1 \end{pmatrix}$$

Q: do Rot and Trans commute?

$$v = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Rot around $\hat{z}$ by $45°$:



$$p \xrightarrow{\text{Tran}} p' \xrightarrow{\text{Rot}} p''$$

$$p \xrightarrow{\text{rot}} p_1 \xrightarrow{\text{trans}} p_2$$

$\longrightarrow$ Rotation & translation don't commute!

$$\mathrm{Rot}(\hat{\omega}, \theta) = \begin{pmatrix} e^{[\hat{\omega}]\theta} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ 0\;0\;0 & 1 \end{pmatrix}.$$

$$\mathrm{Tran}(p) = \begin{pmatrix} 1 & & & \\ & 1 & & p \\ & & 1 & \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We have seen on the example above that

$$\mathrm{Rot}(\hat{\omega},\theta)\,\mathrm{Tran}(p)\neq\mathrm{Tran}(p)\,\mathrm{Rot}(\hat{\omega},\theta)$$

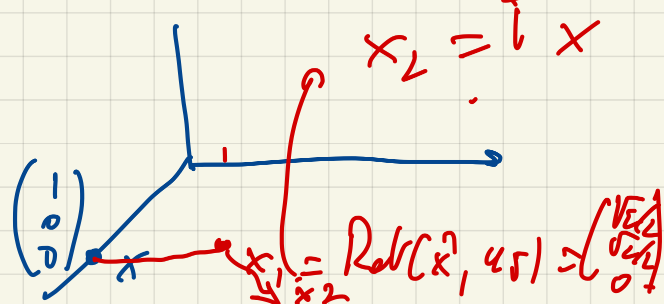
$$\begin{pmatrix} e^{[\hat{\omega}]\theta} & \begin{smallmatrix}0\\0\\0\end{smallmatrix} \\ 0\,0\,0 & 1 \end{pmatrix}\begin{pmatrix} I & p \\ 0\,0\,0 & 1 \end{pmatrix} \qquad \begin{pmatrix} I & p \\ 0\,0\,0 & 1 \end{pmatrix}\begin{pmatrix} e^{[\hat{\omega}]\theta} & \begin{smallmatrix}0\\0\\0\end{smallmatrix} \\ 0\,0\,0 & 1 \end{pmatrix}$$

$$\begin{bmatrix} e^{[\hat{\omega}]\theta} & e^{[\hat{\omega}]\theta}p \\ 0\,0\,0 & 1 \end{bmatrix} \neq \begin{pmatrix} e^{[\hat{\omega}]\theta} & p \\ 0\,0\,0 & 1 \end{pmatrix} \to T$$

$\Rightarrow$ for $T=\begin{pmatrix} R & p \\ 0\,0\,0 & 1 \end{pmatrix}$ = tran by p after rotation by R.

ex: $T=\begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 & 1 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $\quad x=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$T\begin{pmatrix} x \\ 1 \end{pmatrix}=\begin{pmatrix} 1+\sqrt{2}/2 \\ 1+\sqrt{2}/2 \\ 0 \\ 1 \end{pmatrix}$$



- $T_{sb}(t)$: frame that changes with time

$$\frac{d}{dt} T_{sb}(t) = \dot{T}_{sb}(t).$$

<u>recall</u>: $\cdot \; \frac{d}{dt} R_{sb}(t) = [\omega_s] R_{sb}$

$$\rightsquigarrow \underbrace{\dot{R}_{sb} R_{sb}^{-1}}_{\text{"}\dot{R}_{ss}\text{"}} = [\omega_s]$$

and $[\hat{\omega}_b] = \underbrace{R_{sb}^{-1} \dot{R}_{sb}}_{\text{"}\dot{R}_{bb}\text{"}}$

$$T_{sb}^{-1} \dot{T}_{sb} \text{ "=" } \dot{T}_{bb} = [\mathcal{V}_b]$$

$\boxed{\text{TWIST}}$ body twist. $\quad // \; \omega_b$

- $T_{sb}(t) = \begin{pmatrix} R_{sb}(t) & p_s(t) \\ 0\;0\;0 & 1 \end{pmatrix}$

then $\dot{T}_{sb}(t) = \begin{pmatrix} \dot{R}_{sb}(t) & \dot{p} \\ 0\,0\,0 & 0 \end{pmatrix}$

$$T_{sb}^{-1} = \begin{pmatrix} R_{sb}^T & -R_{sb}^T p \\ 0\,0\,0 & 1 \end{pmatrix}$$

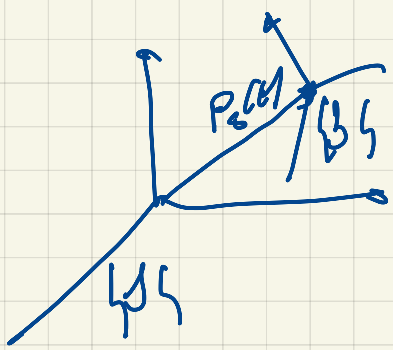
Body twist

$$\rightsquigarrow \quad T_{sb}^{-1} \dot{T}_{sb} = \begin{pmatrix} R_{sb}^T \dot{R}_{sb} & R_{sb}^T \dot{p} \\ 0\,0\,0 & 0 \end{pmatrix} =: [\mathcal{V}_b]$$

$$= \begin{bmatrix} [\omega_b] & \dot{p}_b =: v_b \\ 0\,0\,0 & 0 \end{bmatrix}$$

$R_{sb}^T \dot{p}_s$

$R_{bs} \dot{p}_s = \dot{p}_b$



$p_s(t)$ = coord of center of $\{b\}$ frame expressed in $\{s\}$ frame.

<u>NOTATION</u>

$$\mathcal{V}_b = \begin{pmatrix} \omega_b \\ v_b \end{pmatrix} \begin{matrix} \}3 \\ \}3 \end{matrix} \in \mathbb{R}^6$$

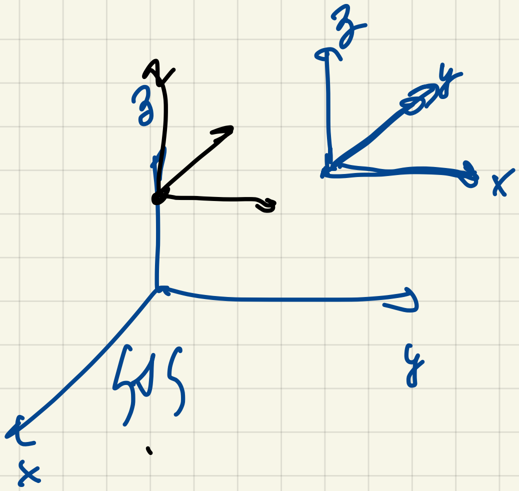
$$[\mathcal{V}_b] := \begin{bmatrix} [\omega_b] & v_b \\ 0\,0\,0 & 0 \end{bmatrix}$$

$$V_b = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} \in \mathbb{R}^6$$

$$[V_b] = \underbrace{\begin{pmatrix} 0 & -w_3 & w_2 & v_1 \\ w_3 & 0 & -w_1 & v_2 \\ -w_2 & w_1 & 0 & v_3 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{u} = \begin{pmatrix} [w] & v \\ 0\,0\,0 & 0 \end{pmatrix}$$

$$V_{\underline{b}\,\underline{b}}^{\underline{s}}$$

s is ref and is "inertial"



$$T_{sb}(t) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & t \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T_{sb}(0) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \times$$

$$\underline{v_b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$$

$$\dot{T}_{sb} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim [V_b] = T_{sb}^{-1} \dot{T}_{sb}$$

$$\underbrace{\begin{pmatrix} 0 & 1 & 0 & -t \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{R^T} \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{d}$$

$$-R^T \begin{pmatrix} 0 \\ t \\ 1 \end{pmatrix} = -\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ t \\ 1 \end{pmatrix} = -\begin{pmatrix} t \\ 0 \\ 1 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{[V_b]}$$

$[w_b]$, v_b

$$\rightarrow w_b = 0$$

$$v_b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

TWISTS. $\to$ FKM
$\rightsquigarrow$ STATICS.

. "velocity" in the space of homogenous transformation.

$\mathcal{V}_b$ = body twist

= velocity of {b}-frame, expressed in coord of {b}-frame, w/ respect to {s}-frame.

$$T_{sb}(t) \rightsquigarrow \frac{d}{dt} T_{sb}(t) = \dot{T}_{sb}$$

$$\left(\frac{d}{dx} x(t) = v(t)\right)$$

$$\rightsquigarrow \quad x^{-1}(t) \frac{d}{dt} x(t)$$

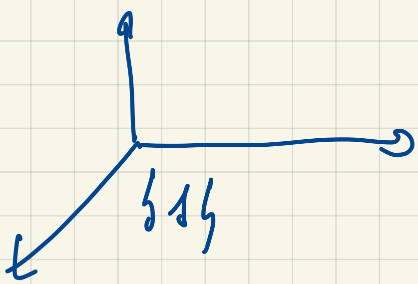
$$\dot{T}_{sb}\, T_{sb}^{-1} \;\text{"="}\; \dot{T}_{sb} = [\mathcal{V}_s]$$

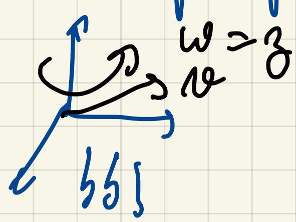
$$\rightsquigarrow \quad \underline{T_{sb}^{-1}\, \dot{T}_{sb}} = [\mathcal{V}_b] \sim \begin{bmatrix} [\omega_b] & v_b \\ 0\;0\;0 & 0 \end{bmatrix}$$

$[\hat{\omega}_s]\theta$

Lie algebra $= se(3) \ni \dot{T}_{sb}$

$T_{sb} \in SE(3)$ Lie group.





$$[\mathcal{V}_b] = \begin{bmatrix} [\omega_b] & v_b \\ 0\;0\;0 & 0 \end{bmatrix}$$

$$\mathcal{V}_b = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} \qquad [\mathcal{V}_b] := \begin{bmatrix} \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} & \begin{matrix} v_1 \\ v_2 \\ v_3 \end{matrix} \\ 0 \quad 0 \quad 0 & 0 \end{bmatrix}.$$

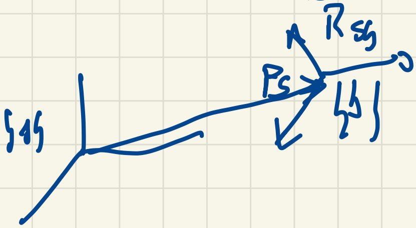
What about $\mathcal{V}$ in $\{s\}$ frame.

$$[\mathcal{V}_s] = \dot{T}_{sb} \, T_{sb}^{-1}$$

$$= \begin{bmatrix} \dot{R}_{sb} & \dot{p}_s \\ 0\,0\,0 & 0 \end{bmatrix} \begin{bmatrix} R_{sb}^T & -R_{sb}^T p_s \\ 0\,0\,0 & 1 \end{bmatrix}$$

$$T_{sb} = \begin{bmatrix} R_{sb} & p_s \\ 0\,0\,0 & 1 \end{bmatrix}$$

$$= \begin{array}{c} \\ 3 \\ 1 \end{array}\begin{bmatrix} \dot{R}_{sb} R_{bs} & -\dot{R}_{sb} R_{bs} p_s + \dot{p}_s \\ 0\,0\,0 & 0 \end{bmatrix}$$

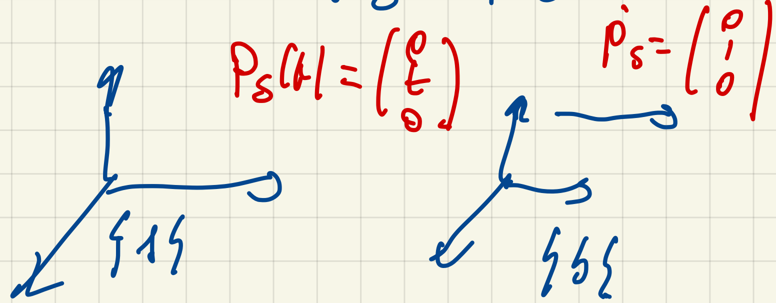


$$= \begin{bmatrix} [\omega_s] & v_s \\ 0\,0\,0 & 0 \end{bmatrix}$$

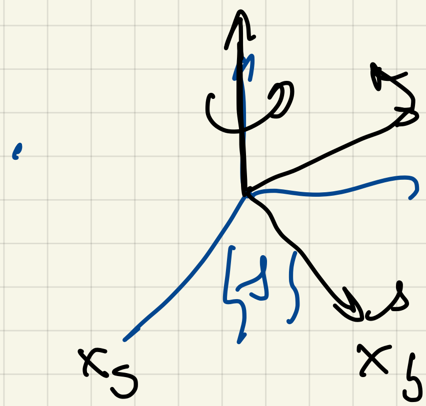
$$-[\omega_s] p_s + \dot{p}_s = -\omega_s \times p + \dot{p}_s =: v_s$$

What is $v_s = -\omega_s \times p_s + \dot{p}_s$?

if $\omega_s = 0$

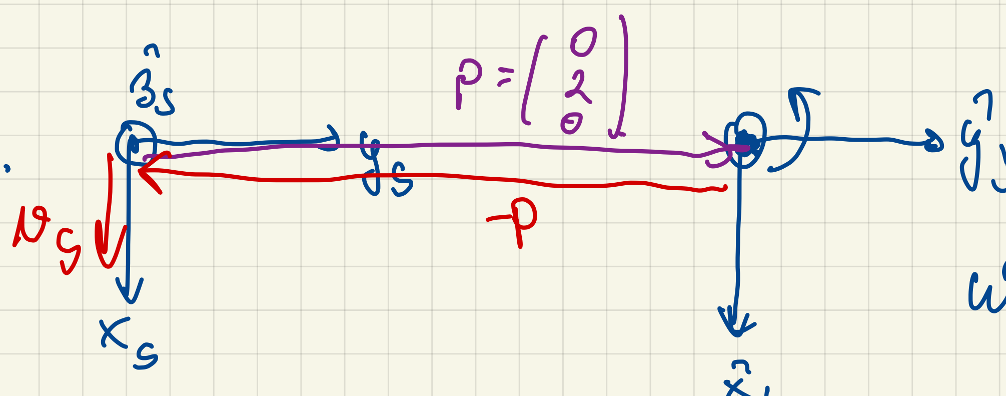


then $v_s = \dot{p}_s$ = velocity of {s} frame



$\omega_s = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ but $p_s = 0$.

$\Rightarrow v_s = 0$

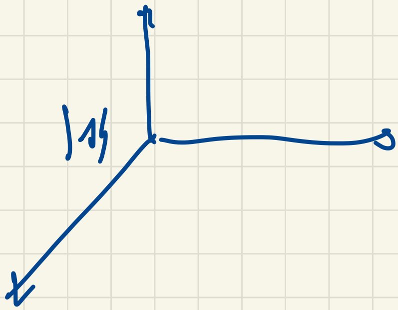


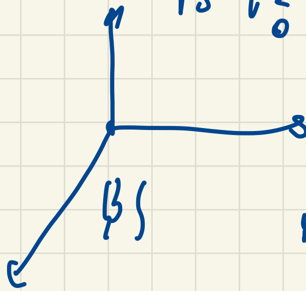
$\dot{p}_s = 0$

$\omega_b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \omega_s$

$$v_s = -\omega_s \times p_s$$
$$= \omega_s \times (-p_s)$$
$$= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

$p_s = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$





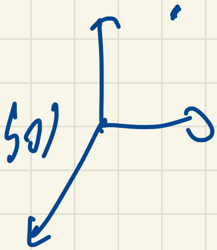
$\dot{p}_s = 0$

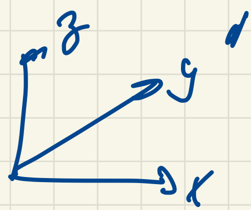
$\omega_b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$v_s = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}$

$v_s = -\omega_s \times p_s + \dot{p}_s$

$= -\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}$





$\omega_b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $p_s = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$

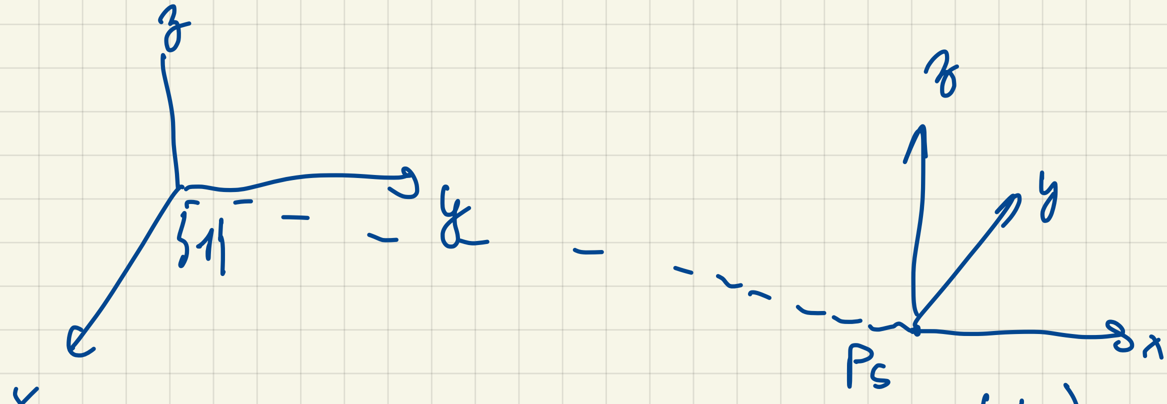
$v_s = 0$

$v_s = -\omega_s \times p_s + \dot{p}_s$

$R_{sb} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$= -\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = 0$

$\omega_s = R_{sb}\,\omega_b$



$$p_s = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \quad w_s = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$v_s =$

$$v_s = -w_s \times p_s + \overset{=0}{\dot{p}_s}$$

$$= -\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$[V_b] = \begin{bmatrix} [w_b] & v_b \\ 0\,0\,0 & 0 \end{bmatrix} \qquad v_b = R_{bs}\dot{p}_s$$

$$[V_s] = \begin{bmatrix} [w_s] & v_s \\ 0\,0\,0 & 0 \end{bmatrix} \qquad v_s = -w_s \times p_s + \dot{p}_s$$

$$[V_b] = T_{sb}^{-1}\dot{T}_{sb} \Rightarrow \dot{T}_{sb} = T_{sb}[V_b]$$

$$[V_s] = \dot{T}_{sb}T_{sb}^{-1} \Rightarrow \dot{T}_{sb} = [V_s]T_{sb}$$

$$\rightarrow \quad T_{sb} = e^{[\mathcal{V}_s]t}\, T_{sb}(0).$$

$$= T_{sb}(0)\, e^{[\mathcal{V}_b]t}.$$