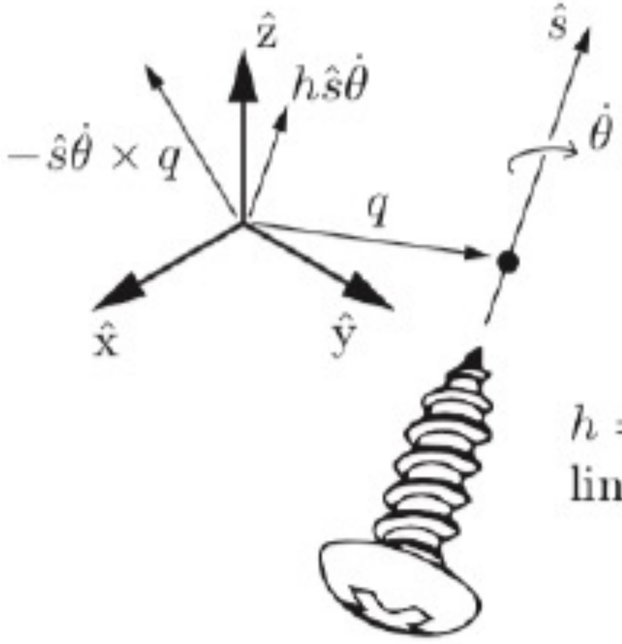
Quick Recap of Past Lectures

1 We can define rotation matrices with a matrix exponential: $R = e^{[\hat{\omega}]\theta}$
Where $e^{[\hat{\omega}]\theta}$ is computed with Rogrigues's Formula
$\hat{\omega}\theta$ are referred to as exponential coordinates

2 For rotations, the only velocities are angular.
For transformations (rotations + translations), we need angular and linear (spatial) velocities!
Twists are the concatenation of ω and v

3 Screw motions give us a generalized concept for defining any motion with a direction and a "screw".
From the information in a twist, we can compute all the components for a screw motion.
Can we connect screws with exponential coordinates to give us a general equation for T?



Screw axis

Given a ref frame, a screw axis $\mathcal{S}$ is defined:

$$\mathcal{S} = \begin{bmatrix} \omega \\ v \end{bmatrix} \in \mathbb{R}^6$$

note: these are overloaded vars!
for screw motions, either $\|\omega\|$ or $\|v\|$ must be 1
→ for twists, ω + v are unconstrained

where either

① $\|\omega\| = 1$

$h=0$ gives pure rot

$v = -\omega \times q + h\omega$

or

② $\|v\| = 1$ and $\omega = 0$

→ translating along v

$$[\mathcal{S}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \in se(3), \quad \mathcal{S}_a = [Ad_{T_{ab}}]\mathcal{S}_b$$

Exponential Coordinates for Rigid Body Motions

- Recall for rotation matrices: $R = e^{[\hat{\omega}]\theta}$
- What are exponential coordinates for T?

  $\mathcal{S}\theta \in \mathbb{R}^6$, where $\mathcal{S}$ is the screw axis and θ is the distance about $\mathcal{S}$ to take frame from I to T

$$T = e^{[\mathcal{S}]\theta} = I + [\mathcal{S}]\theta + [\mathcal{S}]^2 \frac{\theta^2}{2!} + \cdots$$

$$= \begin{bmatrix} e^{[\hat{\omega}]\theta} & G(\theta)v \\ 0 & 1 \end{bmatrix}$$

$$G(\theta) = I\theta + [\omega]\frac{\theta^2}{2!} + \cdots = I\theta + (1 - \cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2$$

Logarithm of rigid body motions

Given $T(R,p) \in SE(3)$, find $\mathcal{S} = [\omega \quad v]^{\mathsf{T}}$ and θ such that:

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \rightarrow [\mathcal{S}]\theta = \begin{bmatrix} [\omega]\theta & v\theta \\ 0 & 0 \end{bmatrix}$$

Define the logarithm of T

1. If $R = I$, then $\omega = 0$ and $v = \frac{p}{\|p\|}$ and $\theta = \|p\|$
2. Find ω, θ using matrix logarithm of R (see earlier slide!)
   Then, $v = G^{-1}(\dot{\theta})p$, where $G^{-1}(\dot{\theta}) = \frac{1}{\theta}I - \frac{1}{2}[\omega] + \left(\frac{1}{\theta} - \frac{1}{2}\cot\left(\frac{\theta}{2}\right)\right)[\omega]^2$

Now we can go back and forth between exponential coordinates and transformations!

Example of Screw Motion for Transformations

Find the pose that is produced by a given screw

Frame 0 is fixed in space and frame 1 is fixed in a body. The current pose of frame 1 in the coordinates of frame 0 is M. The body undergoes a body screw motion with axis $\mathcal{B}$ and magnitude θ. Find the new pose T_{01} of frame 1 in the coordinates of frame 0.

matlab python

```
M = [0.03262475 -0.3
B = [0.00000000; 0.0
theta = 1.29638080;
```

copy this text

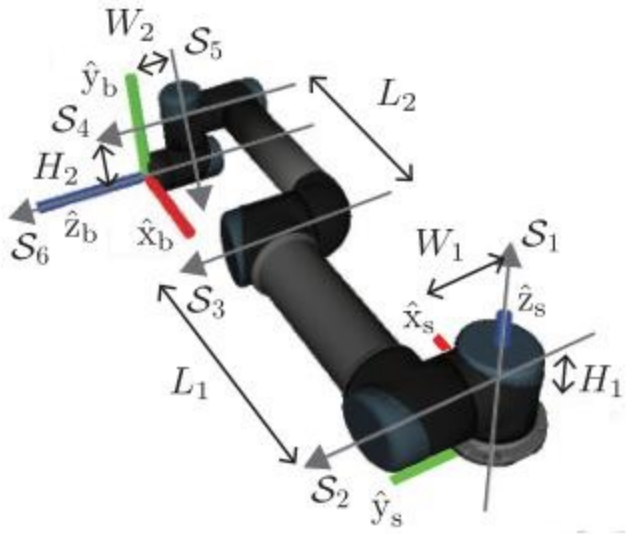
```
[1]: M = [0.03262475 -0.39471442 0.91822445 -0.58853212; -0.21629944 -0.89972977 -0.37907900
     B = [0.00000000; 0.00000000; 0.00000000; -0.67130813; -0.02965809; 0.74058477];
     theta = 1.29638080;

     bb=[0 -B[3] B[2] B[4]
         B[3] 0 -B[1] B[5];
         -B[2] B[1] 0 B[6];
         0 0 0 0];
     tt=exp(bb*theta);
     T=M*tt
```

```
[1]: 4×4 Array{Float64,2}:
       0.0326248  -0.394714   0.918224   0.27982
      -0.216299   -0.89973   -0.379079   0.61705
       0.975782   -0.186244  -0.11473   -0.285795
       0.0         0.0        0.0        1.0
```

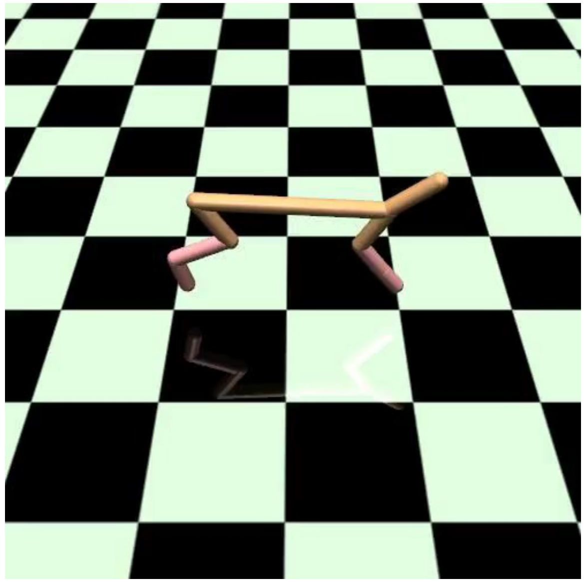
On to Forward Kinematics!

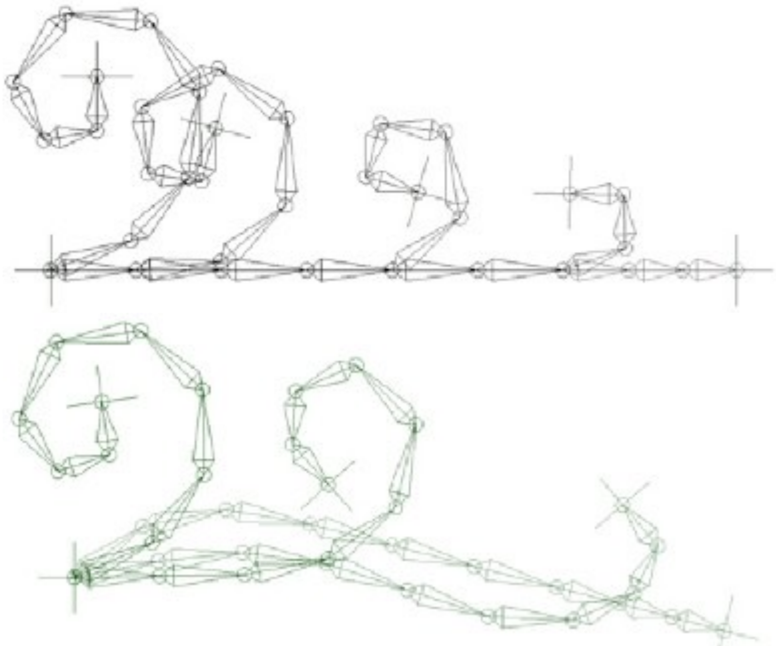
What is Forward Kinematics?

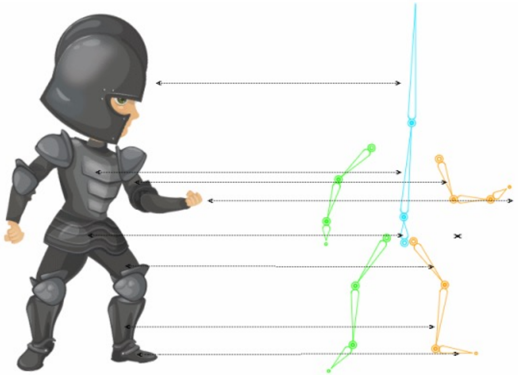
- **Kinematics:** a branch of classical mechanics that describes motion of bodies without considering forces. AKA “the geometry of motion”
- **Forward kinematics:** a specific problem in robotics. Given the individual state of each joint of the robot (in a local frame), what is the position of a given point on the robot in the global frame?

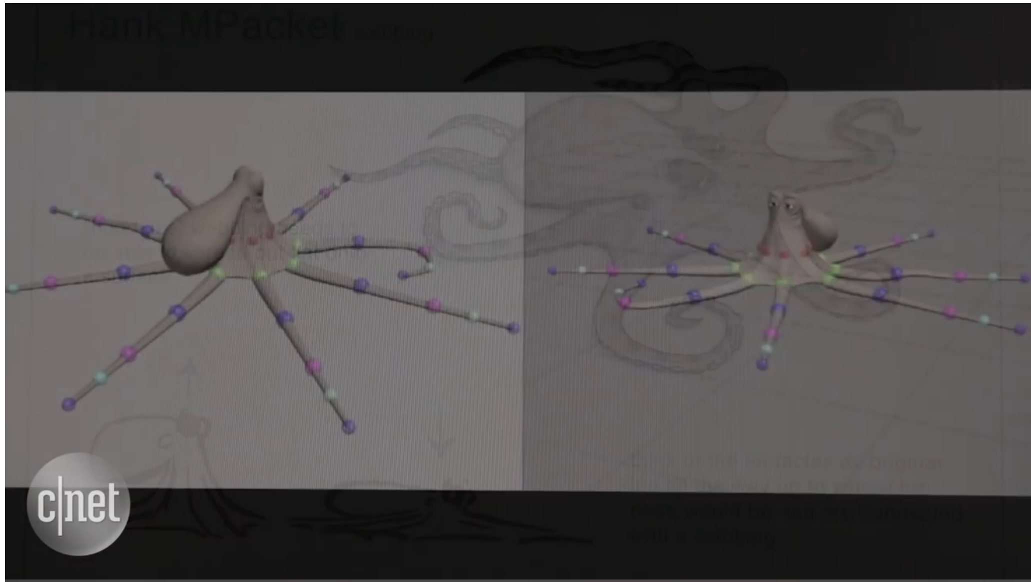


Where is forward kinematics used?

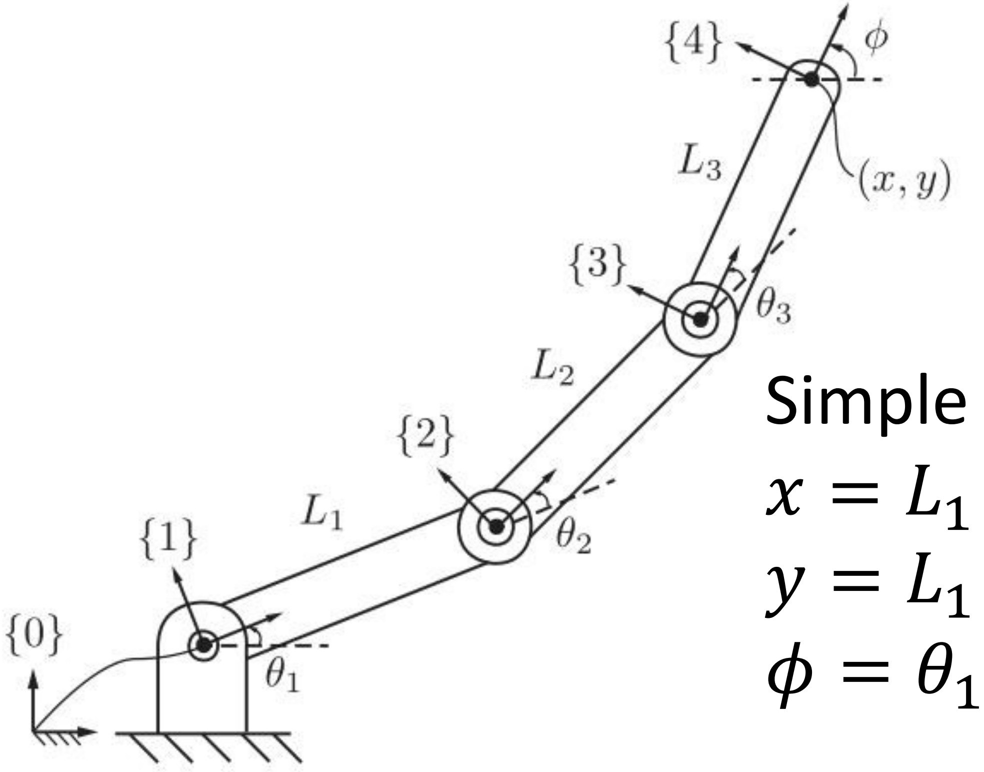








Forward Kinematics of a Simple Chain



Simple Geometry!

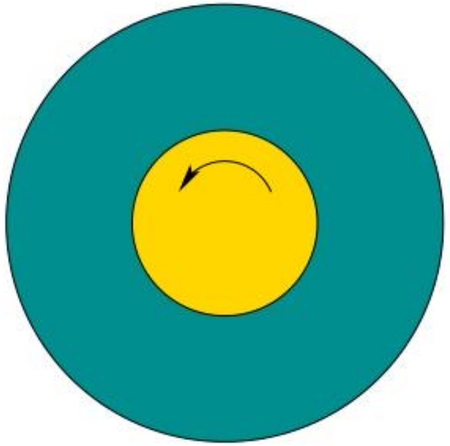
$$x = L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) + L_3 \cos(\theta_1 + \theta_2 + \theta_3)$$
$$y = L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2) + L_3 \sin(\theta_1 + \theta_2 + \theta_3)$$
$$\phi = \theta_1 + \theta_2 + \theta_3$$

Can we use transformations instead?

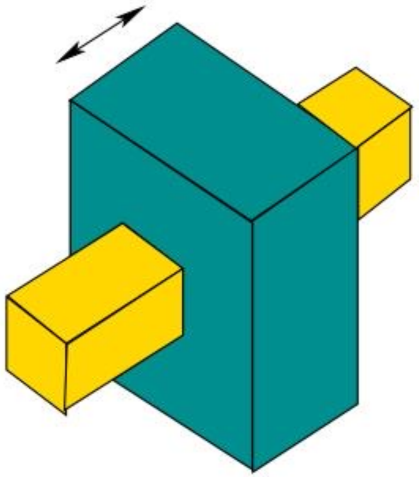
$$T_{04} = T_{01} T_{12} T_{23} T_{34}$$
$$T_{sb} = T_{s1} T_{12} \cdots T_{n-1,n} M$$

Assumptions

- Our robot is a kinematic chain, made of rigid links and movable joints
- No branches or loops
- All joints have one degree of freedom and are revolute or prismatic

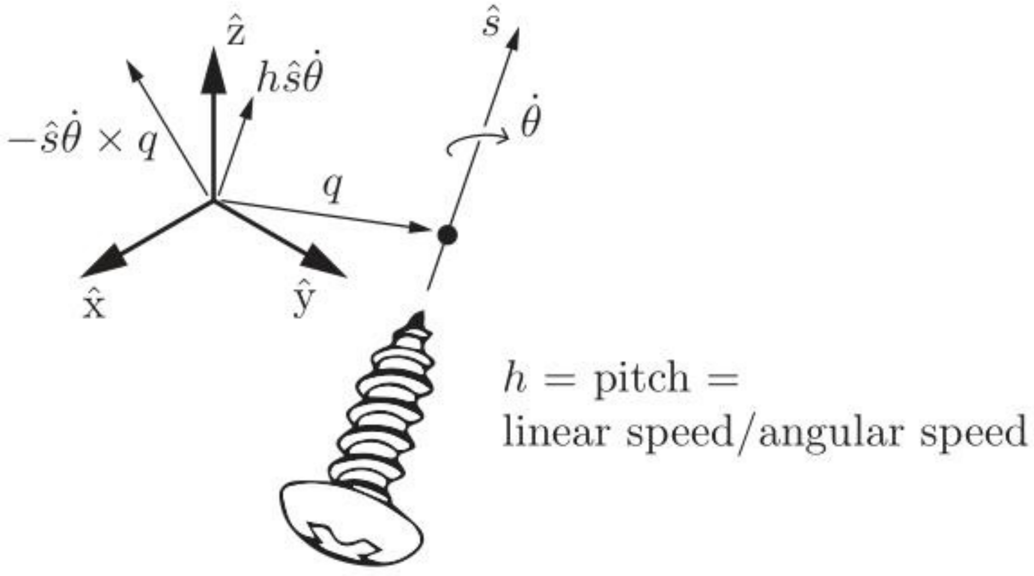


Revolute
1 Degree of Freedom



Prismatic
1 Degree of Freedom

Review of Screw Motions



We derived a representation for the screw axis:

$$\mathcal{S} = \begin{bmatrix} \omega \\ v \end{bmatrix} \in \mathbb{R}^6$$

where either

- $\|\omega\| = 1$
  - where $v = -\omega \times q + h\omega$, where q is the point on the axis of the screw and h is the pitch of the screw
- $\|\omega\| = 0$ and $\|v\| = 1$

Screw Motions as Matrix Exponential

The screw axis $\mathcal{S}_i$ can be expressed in matrix form as:

$$[\mathcal{S}_i] = \begin{bmatrix} [\omega_i] & v \\ 0 & 0 \end{bmatrix} \in se(3)$$

To express a **screw motion** given a screw axis, we use the matrix exponential:

$$e^{[\mathcal{S}]\theta} \in SE(3)$$

Proposition 3.25. *Let $\mathcal{S} = (\omega, v)$ be a screw axis. If $\|\omega\| = 1$ then, for any distance $\theta \in \mathbb{R}$ traveled along the axis,*

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} e^{[\omega]\theta} & \left(I\theta + (1-\cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2\right) v \\ 0 & 1 \end{bmatrix}. \tag{3.88}$$

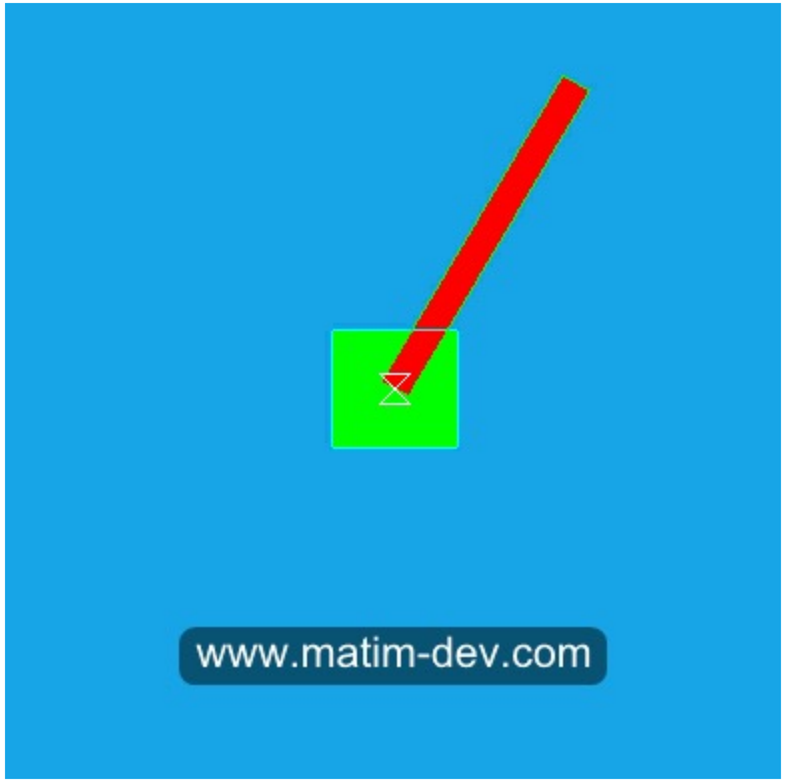
If $\omega = 0$ and $\|v\| = 1$, then

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix}.$$

Modeling Robot Joints as Screw Motions

Case 1: Revolute Joint

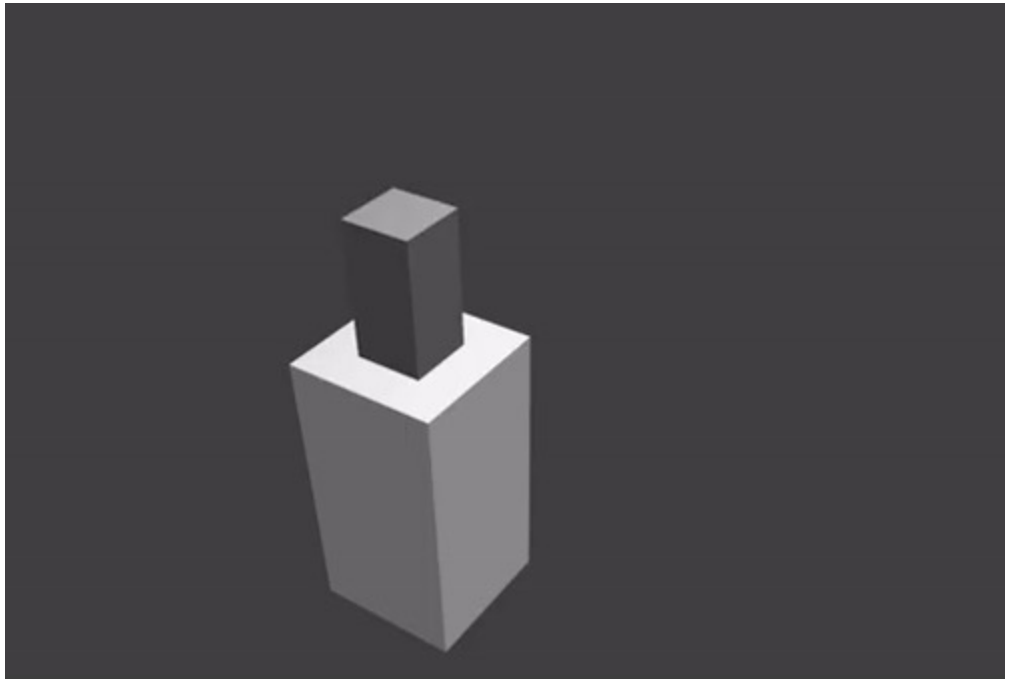
- $\|\omega\| = 1$
- $v = -\omega \times q + h\omega$
- $v = -\omega \times q$



Modeling Robot Joints as Screw Motions

Case 2: Prismatic Joint

- $\|\omega\| = 0$
- $\|v\| = 1$
- Axis of movement defines v



Product of Exponentials Approach

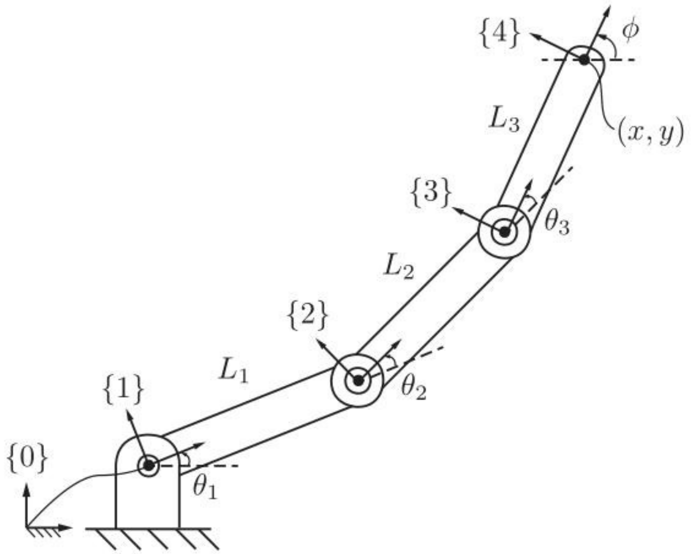


Figure 4.1: Forward kinematics of a 3R planar open chain. For each frame, the x̂- and ŷ-axis is shown; the ẑ-axes are parallel and out of the page.

Let each joint i have a configuration defined by θ_i

Initialization steps:

- Choose a fixed frame $\{s\}$
- Choose an end-effector (tool) frame attached to the robot $\{b\}$
- Put all joints in *zero position*
- Let $M \in SE(3)$ be the configuration of $\{b\}$ in the $\{s\}$ frame when the robot is in the zero position

Product of Exponentials Approach

- Given zero position M
- For each joint i, define the screw axis
- For each motion of a joint, define the screw motion
- These operations compose nicely through multiplication, giving us the Product of Exponentials (PoE) formula!

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} \dots e^{[\mathcal{S}_{n-1}]\theta_{n-1}} e^{[\mathcal{S}_n]\theta_n} M$$

Visualizing the Product of Exponentials

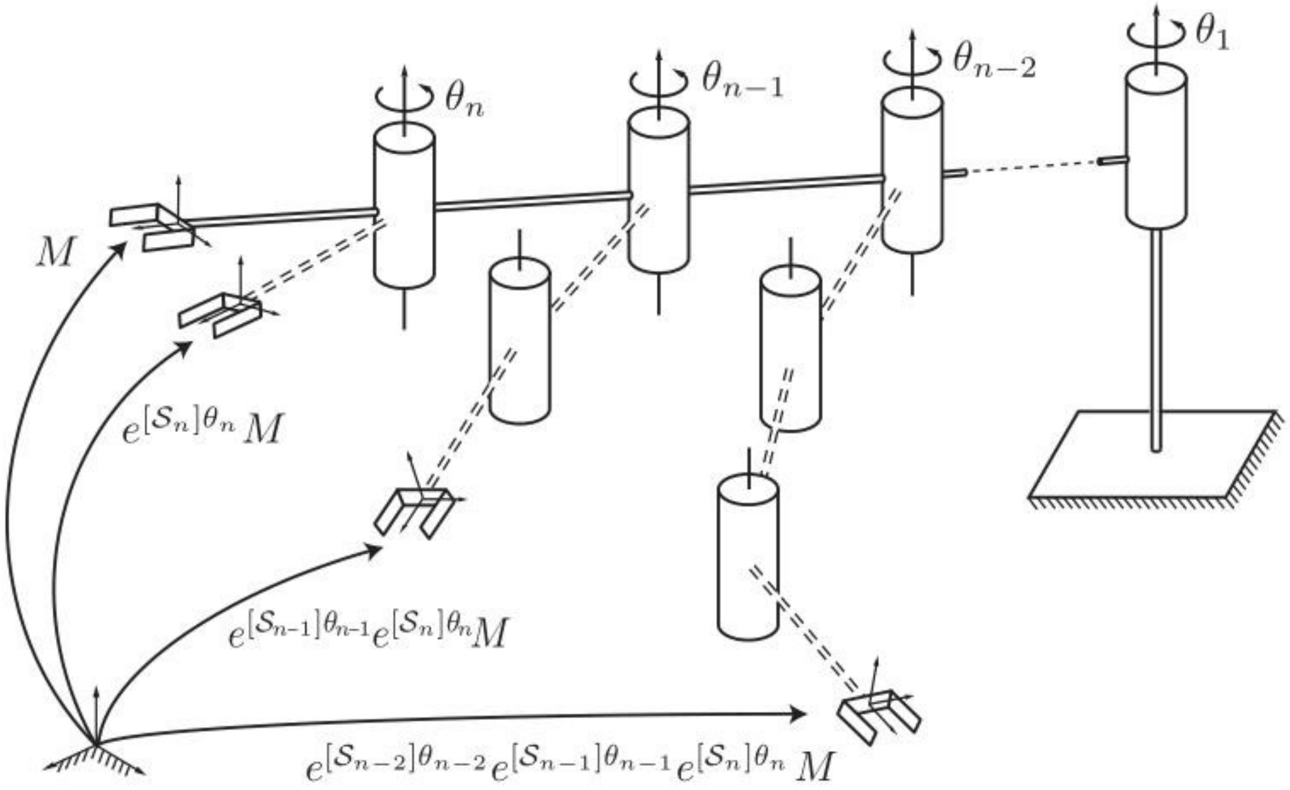


Figure 4.2: Illustration of the PoE formula for an n-link spatial open chain.

Example 1

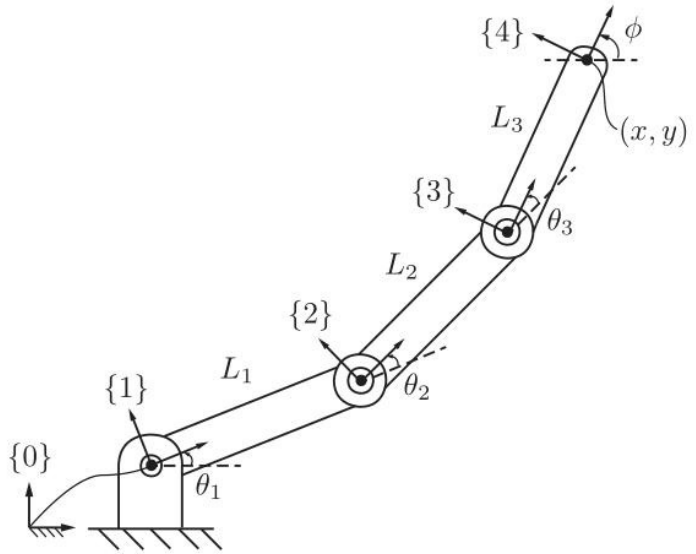


Figure 4.1: Forward kinematics of a 3R planar open chain. For each frame, the $\hat{x}$- and $\hat{y}$-axis is shown; the $\hat{z}$-axes are parallel and out of the page.

Example 1: PoE

Compute $e^{[\mathcal{S}_i]\theta_i}$ for each joint:

$$e^{[\mathcal{S}_i]\theta_i} = \begin{bmatrix} e^{[\omega_i]\theta_i} & (I\theta_i + (1-\cos\theta_i\,)[\omega_i] + (\theta_i - \sin\theta_i)[\omega_i]^2)v_i \\ 0 & 1 \end{bmatrix}$$

and compose with M:

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M$$

Example 2

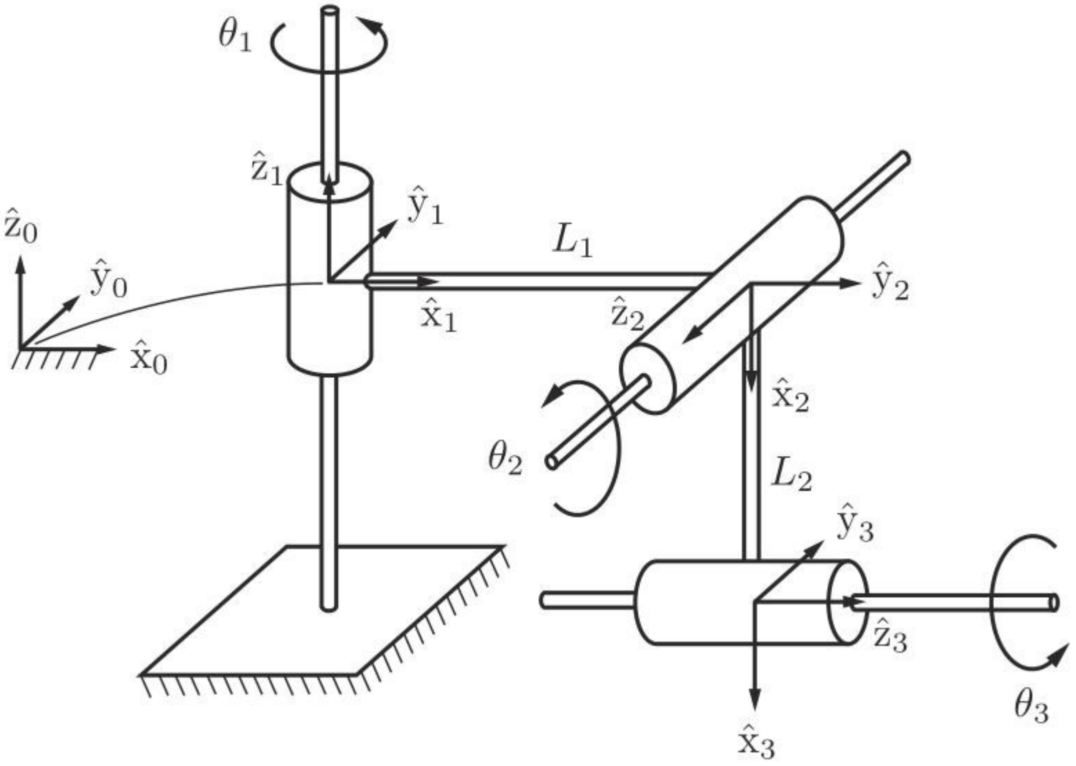
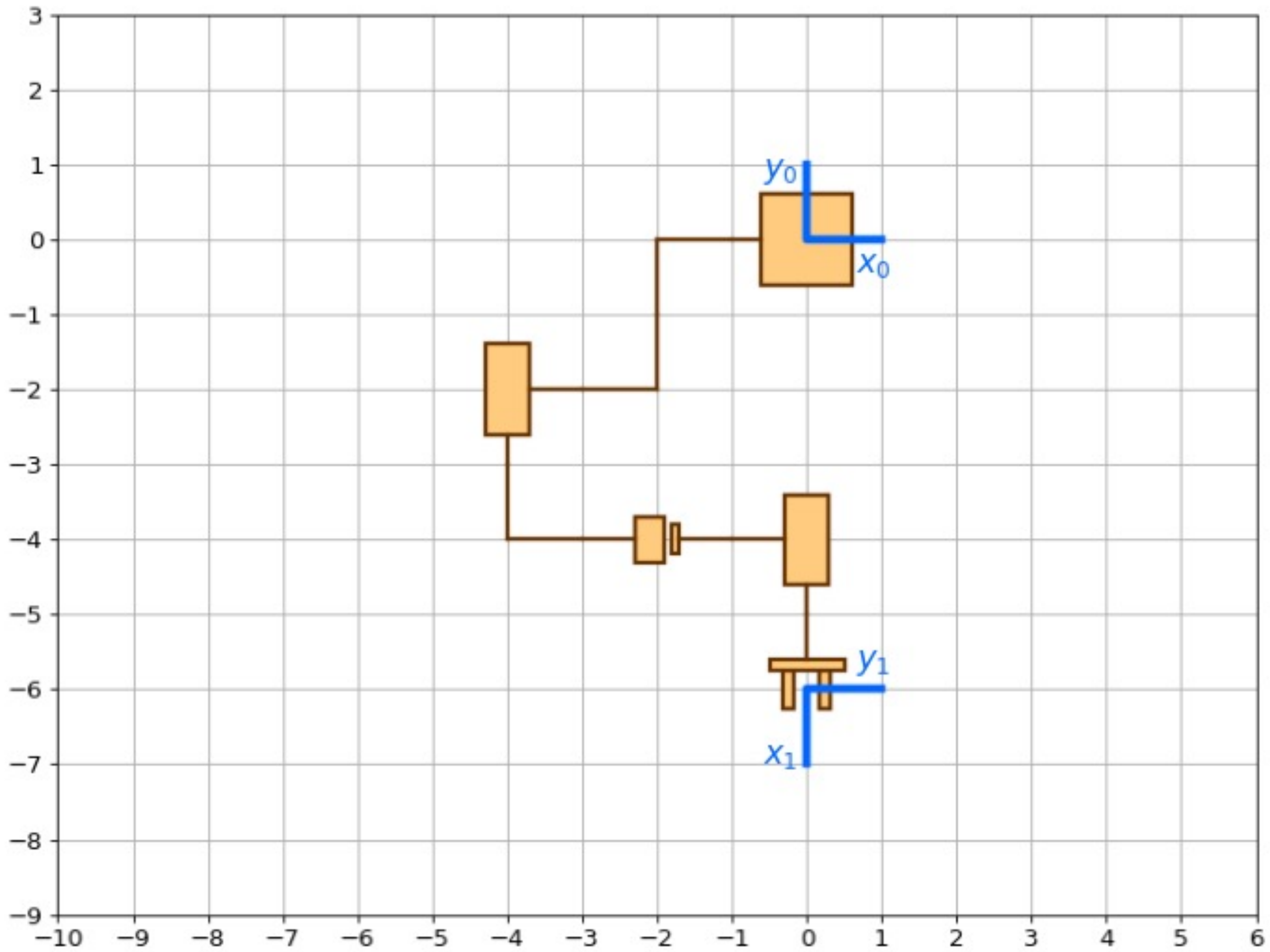
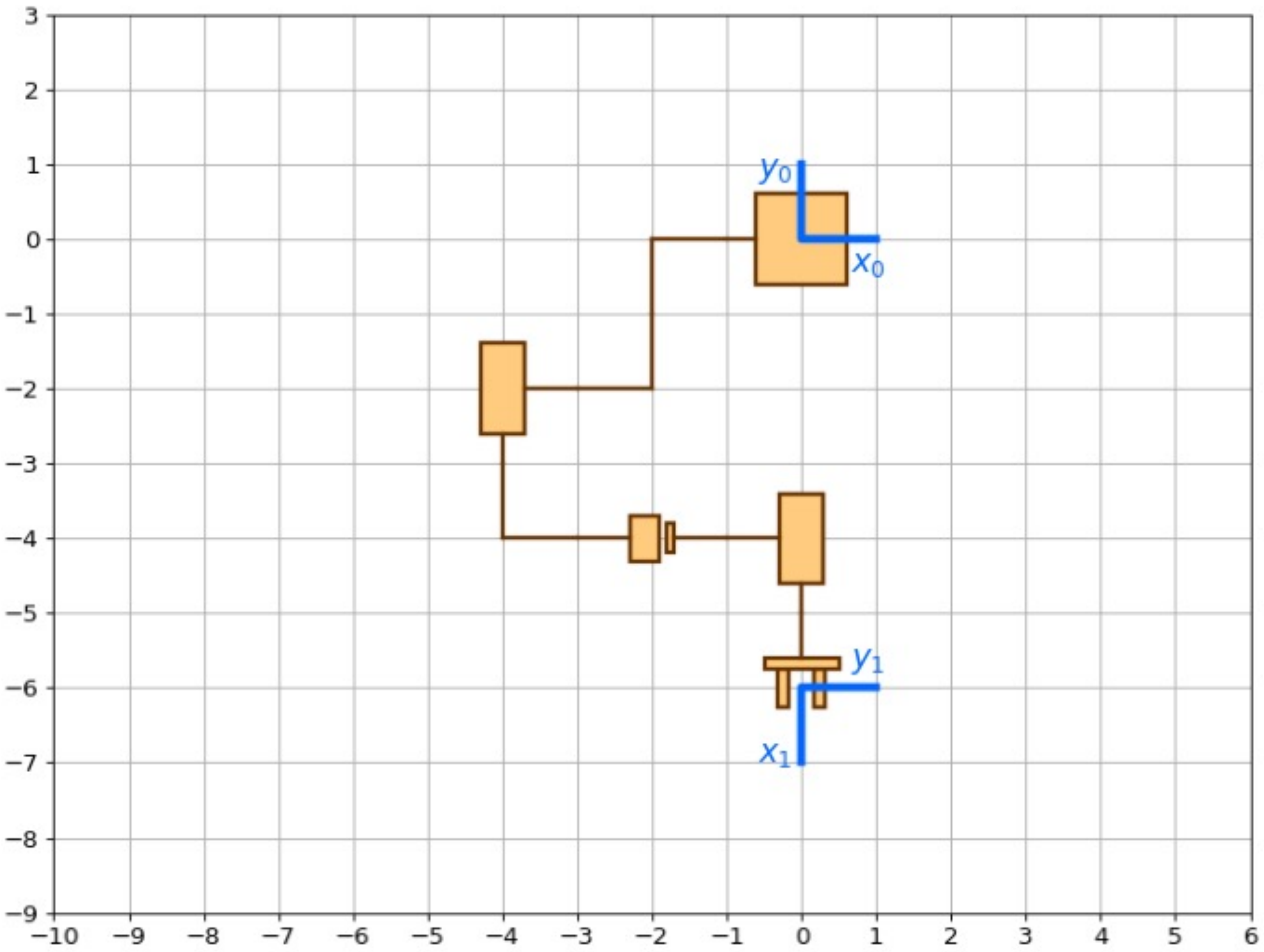


Figure 4.3: A 3R spatial open chain.

Example 3 – How to read this diagram



Example 3



PoE in the End-Effector Frame

- Recall some identities:

$$e^{M^{-1}PM} = M^{-1}e^{P}M \rightarrow e^{P}M = Me^{M^{-1}PM}$$

- Applying this to PoE:

  $T(\theta) = e^{[\mathcal{S}_1]\theta_1} \dots e^{[\mathcal{S}_n]\theta_n} M$

  $T(\theta) = e^{[\mathcal{S}_1]\theta_1} \dots Me^{M^{-1}[\mathcal{S}_n]M\theta_n}$

  $T(\theta) = e^{[\mathcal{S}_1]\theta_1} \dots Me^{M^{-1}[\mathcal{S}_{n-1}]M\theta_{n-1}} e^{M^{-1}[\mathcal{S}_n]M\theta_n}$

  $T(\theta) = Me^{M^{-1}[\mathcal{S}_1]M\theta_1} \dots e^{M^{-1}[\mathcal{S}_{n-1}]M\theta_{n-1}} e^{M^{-1}[\mathcal{S}_n]M\theta_n}$

- What is $M^{-1}[\mathcal{S}_i]M$ (or $[Ad_{M^{-1}}]\mathcal{S}_i$)?
  - Screw axis in the body frame: $[\mathcal{B}_i]$
- Body form of PoE:

$$T(\theta) = Me^{[\mathcal{B}_1]\theta_1} \dots e^{[\mathcal{B}_{n-1}]\theta_{n-1}} e^{[\mathcal{B}_n]\theta_n}$$

Summary

- Used transformations to define spatial velocities as the **body and spatial twist**
- Introduced **screw motions** and the screw interpretation of a twist
- The **screw axis** $\mathcal{S}$ was defined and used to define the **exponential coordinates of homogeneous transformations**
- Learned the basics of **forward kinematics**, which gives us a model for computing position and orientation of the end-effector
- Uses the **product of exponentials formula** to define this transformation as composed matrix multiplications