Credit: Wikipedia

Who was Benjamin Olinde Rodrigues?

- A mathematician who turned to finance for employment
  - PhD in math and became a banker
- His dissertation introduced the Rodrigues formula
  - Formerly known as the Ivory–Jacobi formula, as it was independently introduced by Olinde Rodrigues (1816), Sir James Ivory (1824), and **Carl Gustav Jacobi (1827)**

Angular Velocities in Reference Frame (1)

let $R(t)$ describe orientation of a body w.r.t. $\{s\}$

$$R(t) = \begin{array}{c} \begin{bmatrix} | & | & | \\ r_1(t) & r_2(t) & r_3(t) \\ | & | & | \end{bmatrix} \\ \begin{array}{ccc} \hat{x} \text{ in } \{s\} & \hat{y} & \hat{z} \end{array} \end{array}$$

let $\omega_s = \omega$ be angular velocity in $\underline{\{s\}}$

$$\dot{r}_i = \omega_s \times r_i \quad \forall i \longrightarrow \dot{R} = [\underline{\omega \times} r_1 \quad \underline{\omega \times r_2} \quad \underline{\omega \times r_3}]$$

$$= \underbrace{\omega \times R}$$

Angular Velocities in Reference Frame (2)

for each ω, define a unique, $\underbrace{\text{skew-symmetric}}_{A=-A^T}$ matrix

$$\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \longleftrightarrow \underbrace{\begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}}_{} := [\omega] \in so(3)$$

$$\dot{R} = \omega \times R = [\omega]R \longleftrightarrow [\omega] = \dot{R}R^{-1} = \dot{R}R^T$$

Introducing
Exponential Coordinates

Recall the Matrix Exponential

If $\dot{x}(t) = Ax(t), x(0) = x_0$, where $x \in \mathbb{R}^n, A \in \mathbb{R}^{n\times n}$, the solution to the ODE is:

$$x(t) = e^{At}x_0$$

The matrix exponential / the exponential function can be expanded:

$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots$$

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Properties:

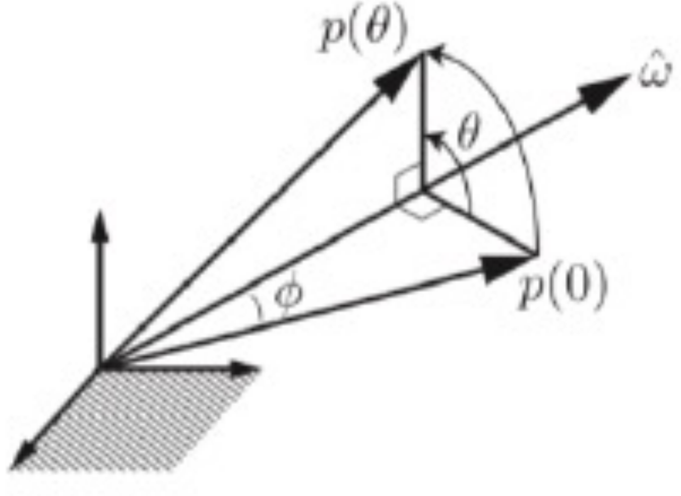
1. $\frac{d}{dt}(e^{At}) = Ae^{At} = e^{At}A$
2. If $A = PDP^{-1}$, then $e^{At} = Pe^{Dt}P^{-1}$
3. If $AB = BA$, then $e^A e^B = e^B e^A = e^{A+B}$
4. Invertible: $(e^{At})^{-1} = e^{-At}$

Exponential Coordinate Representation

- Instead representing orientation as a rotation matrix, we introduce a three-parameter representation: **exponential coordinates**
- Recall: $\widehat{\omega}$ rotation axis, θ angle of rotation
- Then $\widehat{\omega}\theta \in \mathbb{R}^3$gives the exponential coordinate representation

- A new interpretation for a frame coincident with $\{s\}$:
  - A rotation for 1 second around $\widehat{\omega}$ at angular velocity θ, then the resulting frame is R
  - A rotation for θ seconds around $\widehat{\omega}$ at angular velocity 1, then the resulting frame is R

Exponential Coordinates of Rotations



A few tricks…

Matrix Logarithm

- Given rotation matrix R, we need to take the *logarithm* to find the exponential coordinates:

exp:	$[\hat{\omega}]\theta \in so(3)$	$\rightarrow R \in SO(3)$
log:	$R \in SO(3)$	$\rightarrow [\hat{\omega}]\theta \in so(3)$

Matrix Logarithm

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$$\begin{array}{lll} \text{exp:} & [\hat{\omega}]\theta \in so(3) & \to R \in SO(3) \\ \text{log:} & R \in SO(3) & \to [\hat{\omega}]\theta \in so(3) \end{array}$$

- If we expand Rodrigues formula:

$$\mathrm{Rot}(\hat{\omega}, \theta) = I + \sin\theta\,[\hat{\omega}] + (1 - \cos\theta)[\hat{\omega}]^2$$

$$\mathrm{Rot}(\hat{\omega}, \theta) = \begin{bmatrix} c_\theta + \hat{\omega}_1^2(1 - c_\theta) & \hat{\omega}_1\hat{\omega}_2(1 - c_\theta) - \hat{\omega}_3 s_\theta & \hat{\omega}_1\hat{\omega}_3(1 - c_\theta) + \hat{\omega}_2 s_\theta \\ \hat{\omega}_1\hat{\omega}_2(1 - c_\theta) + \hat{\omega}_3 s_\theta & c_\theta + \hat{\omega}_2^2(1 - c_\theta) & \hat{\omega}_2\hat{\omega}_3(1 - c_\theta) - \hat{\omega}_1 s_\theta \\ \hat{\omega}_1\hat{\omega}_3(1 - c_\theta) - \hat{\omega}_2 s_\theta & \hat{\omega}_2\hat{\omega}_3(1 - c_\theta) + \hat{\omega}_1 s_\theta & c_\theta + \hat{\omega}_3^2(1 - c_\theta) \end{bmatrix}$$

Matrix Logarithm Method

- If we take the trace of the matrix, we can solve for θ:
  - $tr(R) \coloneqq r_{11} + r_{22} + r_{33} = 1 + 2\cos\theta$

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- If we compute $R - R^{\top}$, we get:

$$\left.\begin{aligned} r_{32} - r_{23} &= 2\hat{\omega}_1 \sin\theta \\ r_{13} - r_{31} &= 2\hat{\omega}_2 \sin\theta \\ r_{21} - r_{12} &= 2\hat{\omega}_3 \sin\theta \end{aligned}\right\} \quad [\hat{\omega}] = \frac{1}{2\sin\theta}(R - R^{\top})$$

Matrix Logarithm Method

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- If we compute $R^\top - R$, we get:

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- But what if $\sin\theta = 0$? (called singularities)
  - If $\theta = 2k\pi$, we have rotated by 360 degrees
  - If $\theta = (2k+1)\pi$, then Rodrigues formula is $R = I + 2[\hat{\omega}]^2$, which gives: $\hat{\omega}_i = \pm\sqrt{\frac{r_{ii}+1}{2}}$ and $2\hat{\omega}_i\hat{\omega}_j = r_{ij}$ for $i \neq j$

On to Homogeneous Transformations!

Homogeneous Transformations: SE(3)

The **special Euclidean group** $\boldsymbol{SE(3)}$ is the set of 4×4 matrices of the form:

$$T = T(R, p) = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$$

Homogeneous Transformations: SE(3)

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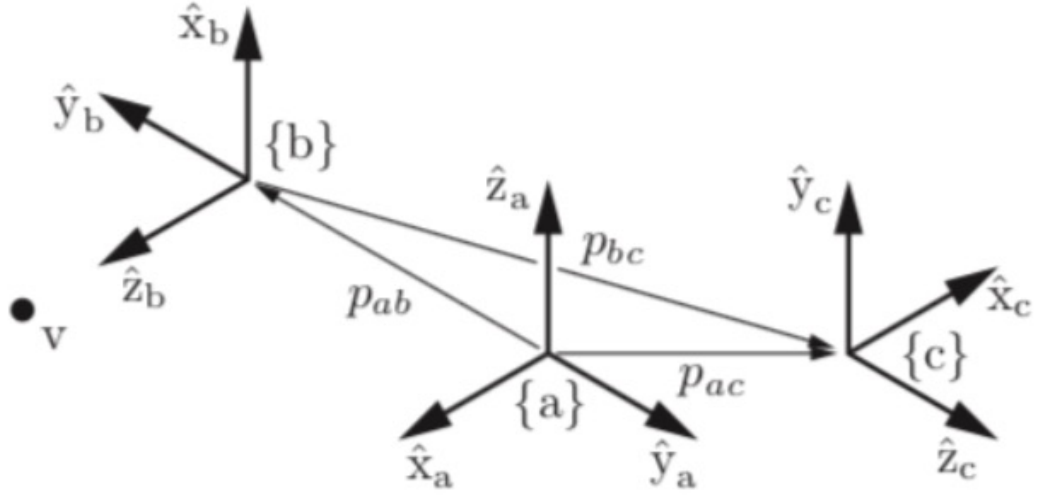
$$T = T(R, p) = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$$

- The inverse of T is $T^{-1} = \begin{bmatrix} R^\top & -R^\top p \\ 0 & 1 \end{bmatrix} \in SE(3)$
- If T_1 and $T_2 \in SE(3)$, then $T_1 T_2 \in SE(3)$
- T satisfies: $\|Tx - Ty\| = \|x - y\|$ and

$$(Tx - Tz)^\top (Ty - Tz) = (x - z)^\top (y - z)$$

Representing a configuration with SE(3)



Each frame can represent a body frame in a multi-link mechanism

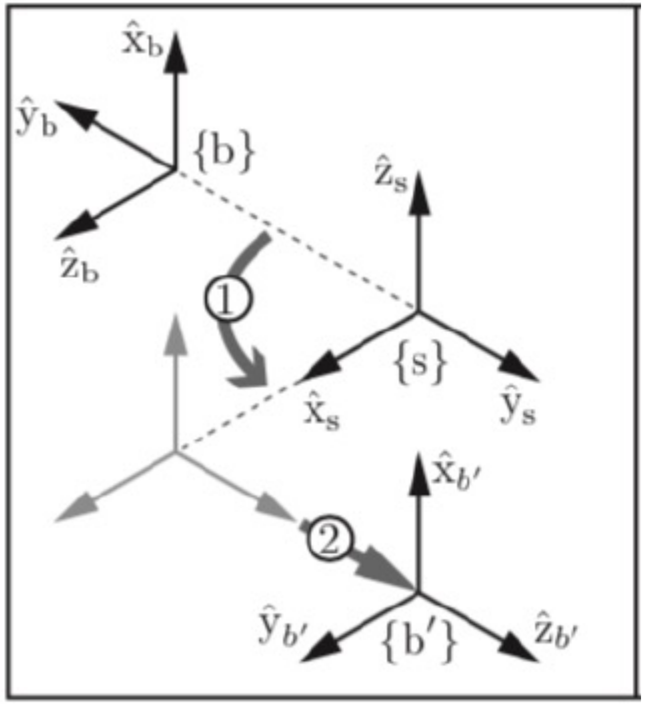
As before:

$$T_{ab}T_{bc} = T_{ac}$$

$$T_{ab}v_b = v_a$$

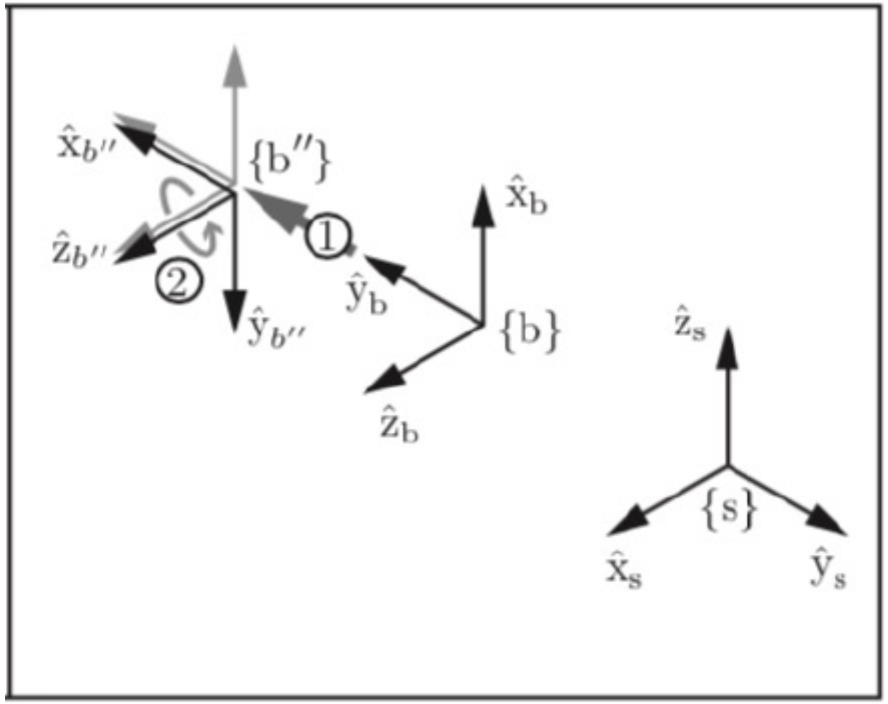
Displacing Frames (1)

Example Displacement (1)



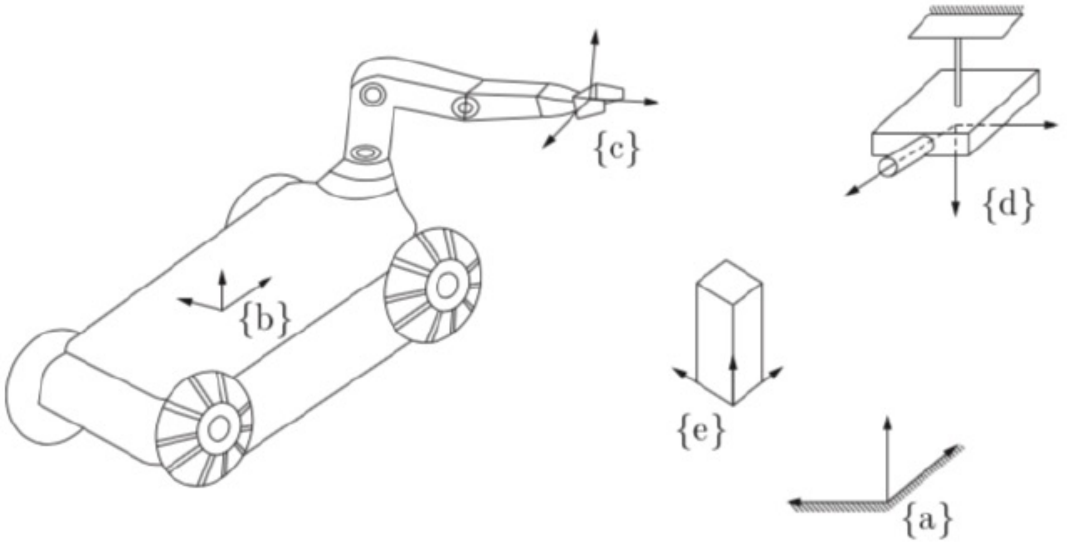
Displacing Frames (2)

Example Displacement (2)



Mobile arm example

- Robot arm mounted on wheeled platform. Camera fixed to ceiling.
- {b} is body frame, {c} end-effector frame, {e} frame of object, and {a} is fixed frame.
- We assume the camera position and orientation in {a} is given.
- From camera measurements, you can evaluate the position and orientation of the body and the object in the camera frame.
- Since we designed our robot and have joint-angle estimates, we can obtain the end-effector position and orientation in the body frame.
- To pick up the object, we need the object position and orientation in the frame of our end-effector.



Summary

- Introduced **exponential coordinates** that allow us to parameterize rotations by the rotation axis $\widehat{\omega}$ and the angle of rotation θ
- Walked through the **matrix logarithm** method, which tells us how to recover the exponential coordinates from a rotation matrix
- Started discussing **homogeneous transformations** where we not only look at rotations, but also translations in a single operation