Credit: Wikipedia

Who was Benjamin Olinde Rodrigues?

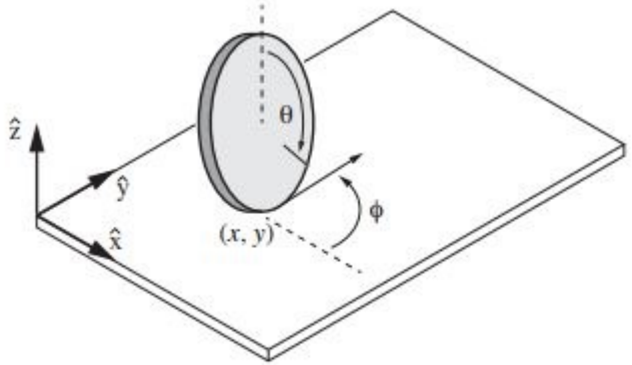
- A mathematician who turned to finance for employment
  - PhD in math and became a banker
- His dissertation introduced the Rodrigues formula
  - Formerly known as the Ivory–Jacobi formula, as it was independently introduced by Olinde Rodrigues (1816), Sir James Ivory (1824), and **Carl Gustav Jacobi (1827)**

Types of Constraints

Types of Constraints

- Suppose we are given Pfaffian constraints: $A(\theta)\dot{\theta} = 0$
- If we can find a function g such that $\frac{\partial g}{\partial \theta} = A$, the (integrable) constraints g are **holonomic**
    - Why? If such g exists, the constraints $A(\theta)\dot{\theta} = 0$ are the same as the constraints on the position variables $g(\theta)$
- If no such g exists, the constraints are called **non-holonomic**

Rolling Penny Example



Summary

- Introduced fundamental robotics definitions like **configuration space, task space,** and **workspace**
- Learned methods for determining the number of **degrees of freedom** for a rigid body and robot mechanism
- Discussed **holonomic and non-holonomic constraints** which may affect the configuration space and mobility of a robot

Rigid Body Motions

Frames of Reference

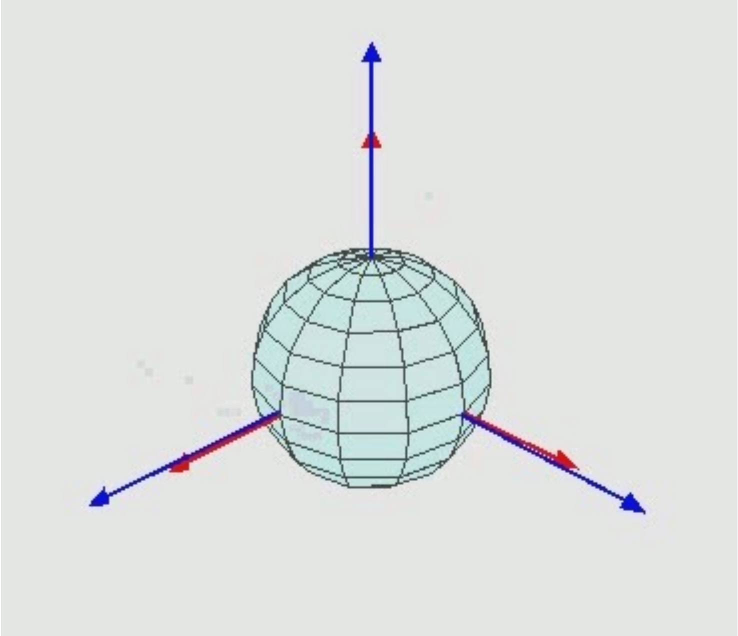
Rotation Matrices (1)

- 2D rotations:

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

- 3D rotations about a main axis:

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$



Credit: Wikipedia

Rotation Matrices (2)

- We use the convention R_{ab} for the matrix whose **columns are coordinates of the frame $\boldsymbol{b}$ expressed in the frame $\boldsymbol{a}$**
- We have the following laws:

$$R_{ab}R_{bc} = R_{ac} \text{ and } R_{ab}^{-1} = R_{ba}$$

- If the vector p_a is the vector p expressed in frame a, then

$$R_{ba}p_a = p_b$$

- If we rotate a vector v around the $\widehat{\omega}$ axis by angle θ, we say $\mathrm{Rot}(\widehat{\omega}, \theta)v$
  - For a rotation about the x-axis: $\mathrm{Rot}(\hat{x}, \theta)v = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} v$
  - Rotations about the y and z axis follow

Special Orthogonal Groups

$\boldsymbol{SO(3)}$, the group that represents all spatial rotation matrices, is the set of all 3×3 real matrices R that satisfy

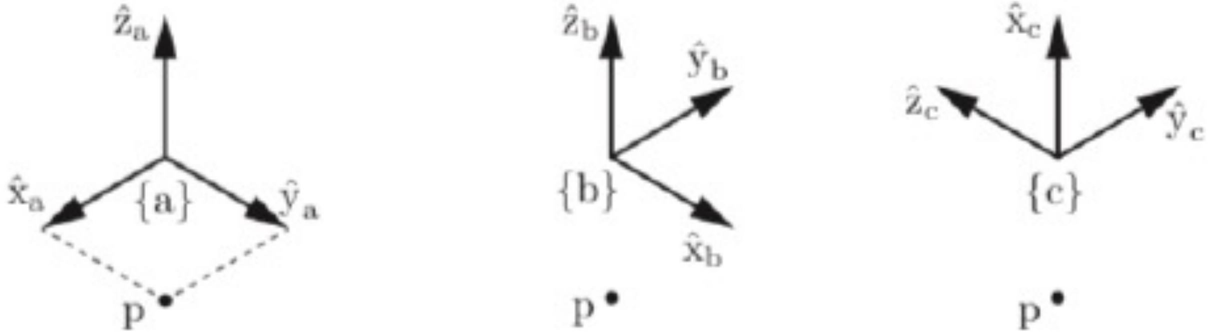
1. $R^\top R = I$
2. $\det R = 1$

- Note that (2) implies that both (1) holds and that we obey the righthand rule

Similarly, $\boldsymbol{SO(2)}$, the group describes all planar rotation matrices, is the set of all 2×2 real matrices R, satisfying the same conditions.

Changing frames

Frames in 3D



$\hat{z}_a$
$\hat{x}_a$
$\{a\}$
$\hat{y}_a$
p
$\hat{z}_b$
$\hat{y}_b$
$\{b\}$
$\hat{x}_b$
p
$\hat{x}_c$
$\hat{z}_c$
$\hat{y}_c$
$\{c\}$
p

Affine Transformations (more next lecture!)

Rigid Body Motions

Recall the cross product

- The cross-product is defined as
$$a \times b = \|a\| \|b\| \sin\theta \, \hat{n}$$

- In coordinates:
$$a \times b = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

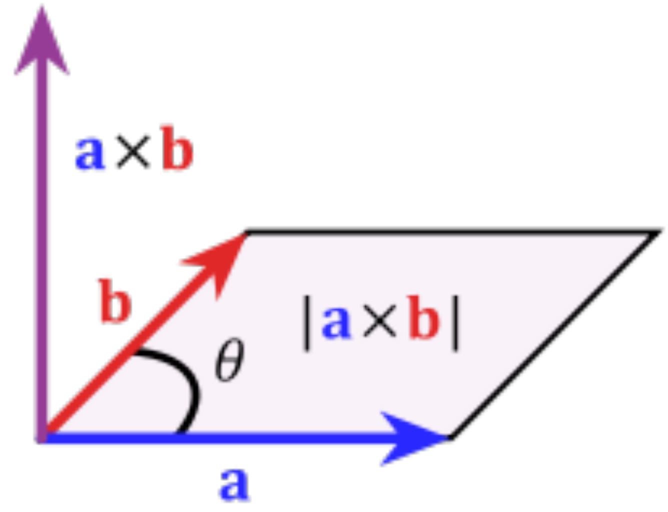
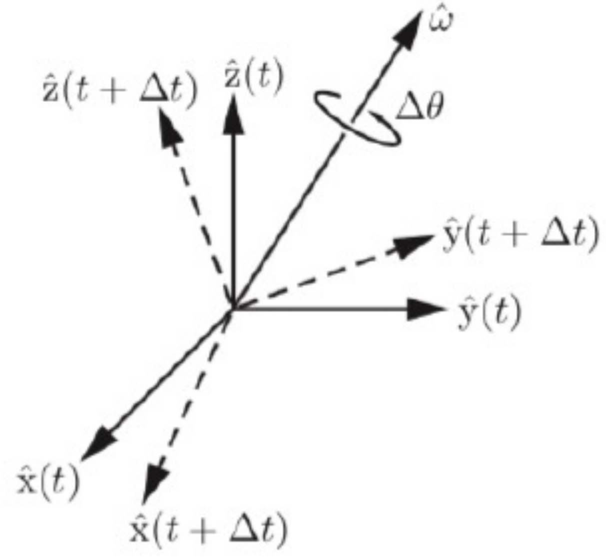
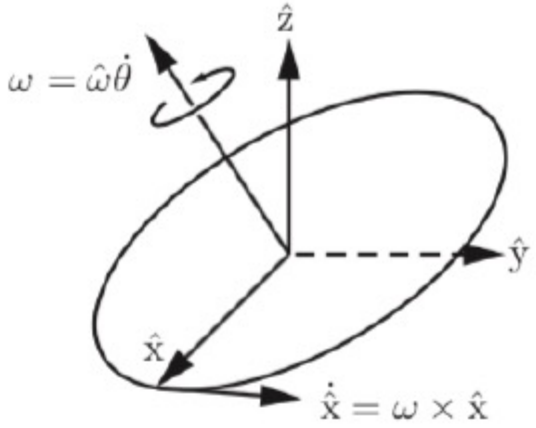


Image Credit: Wikipedia

Angular Velocities





Angular Velocities in Reference Frame

Some useful properties and relations

Given any $\omega \in \mathbb{R}^3$ and $R \in SO(3)$, the following holds: $R[\omega]R^\top = [R\omega]$

Now recall: $[\omega_s] = \dot{R}R^\top$

If R is R_{sb}, we have that $\omega_s = R\omega_b \Leftrightarrow \omega_b = R^\top \omega_s$

$$[\omega_b] = [R^\top \omega_s] = R^\top (\dot{R}R^\top) R = R^\top \dot{R} = R^{-1}\dot{R}$$

This gives us: $[\omega_s] = \dot{R}R^\top$ and $[\omega_b] = R^\top \dot{R}$

Summary

- Discussed **holonomic and non-holonomic constraints** which may affect the configuration space and mobility of a robot
- Formally defined **frames of reference** and **SO(3),** and built intuition for rigid body *motion*
- Defined **bracket notation** $[\boldsymbol{\omega}]$**,** a skew-symmetric representation of our angular velocity