

Lecture 05: Exponential Coordinates & Homogeneous Transformations

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September 10, 2024

Administrivia

- HW2 is due Friday
 - Note that there will be weekly homework
- **Just this week:** Prof. DC OH moved to Thursday at 2pm



Credit: Wikipedia

Who was Benjamin Olinde Rodrigues?

- A mathematician who turned to finance for employment
 - PhD in math and became a banker
- His dissertation introduced the Rodrigues formula
 - Formerly known as the Ivory–Jacobi formula, as it was independently introduced by Olinde Rodrigues (1816), Sir James Ivory (1824), and **Carl Gustav Jacobi (1827)**

Angular Velocities in Reference Frame (1)

let $R(t)$ describe orientation of a body w.r.t. $\{s\}$

$$R(t) = \begin{bmatrix} \underset{\substack{\uparrow \\ \hat{x} \text{ in } \{s\}}}{r_1'(t)} & \underset{\substack{\uparrow \\ \hat{y}}}{r_2'(t)} & \underset{\substack{\uparrow \\ \hat{z}}}{r_3'(t)} \end{bmatrix}$$

let $\omega_s = \omega$ be angular velocity in $\{s\}$

$$\dot{r}_i = \omega_s \times r_i \quad \forall i \leadsto \dot{R} = \begin{bmatrix} \underline{\omega \times r_1} & \underline{\omega \times r_2} & \underline{\omega \times r_3} \end{bmatrix} \\ = \underline{\omega \times R}$$

Angular Velocities in Reference Frame (2)

for each w , define a unique,
skew-symmetric matrix

$$w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

$$A = -A^T$$



$$\begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix} \stackrel{e_{so(3)}}{:=} \underline{[w]}$$

$$\dot{R} = w \times R = [w]R \quad \longleftrightarrow \quad [w] = \dot{R}R^{-1} = \underline{\underline{\dot{R}R^T}}$$

Introducing Exponential Coordinates

Recall the Matrix Exponential

If $\dot{x}(t) = Ax(t)$, $x(0) = x_0$, where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$, the solution to the ODE is:

$$x(t) = e^{At}x_0$$

The matrix exponential / the exponential function can be expanded:

$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \dots$$

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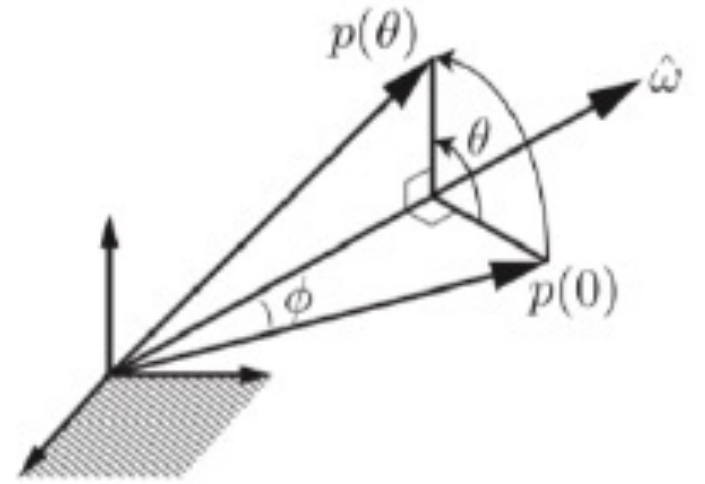
Properties:

1. $\frac{d}{dt}(e^{At}) = Ae^{At} = e^{At}A$
2. If $A = PDP^{-1}$, then $e^{At} = Pe^{Dt}P^{-1}$
3. If $AB = BA$, then $e^Ae^B = e^Be^A = e^{A+B}$
4. Invertible: $(e^{At})^{-1} = e^{-At}$

Exponential Coordinate Representation

- Instead representing orientation as a rotation matrix, we introduce a three-parameter representation: **exponential coordinates**
- Recall: $\hat{\omega}$ rotation axis, θ angle of rotation
- Then $\hat{\omega}\theta \in \mathbb{R}^3$ gives the exponential coordinate representation
- A new interpretation for a frame coincident with $\{s\}$:
 - A rotation for 1 second around $\hat{\omega}$ at angular velocity θ , then the resulting frame is R
 - A rotation for θ seconds around $\hat{\omega}$ at angular velocity 1, then the resulting frame is R

Exponential Coordinates of Rotations



A few tricks...

Matrix Logarithm

- Given rotation matrix R , we need to take the *logarithm* to find the exponential coordinates:

$$\text{exp:} \quad [\hat{\omega}]\theta \in so(3) \rightarrow R \in SO(3)$$

$$\text{log:} \quad R \in SO(3) \rightarrow [\hat{\omega}]\theta \in so(3)$$

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- If we expand Rodrigues formula:

$$\text{Rot}(\hat{\omega}, \theta) = I + \sin \theta [\hat{\omega}] + (1 - \cos \theta)[\hat{\omega}]^2$$

$$\begin{aligned} &\text{Rot}(\hat{\omega}, \theta) \\ &= \begin{bmatrix} c_\theta + \hat{\omega}_1^2(1 - c_\theta) & \hat{\omega}_1\hat{\omega}_2(1 - c_\theta) - \hat{\omega}_3s_\theta & \hat{\omega}_1\hat{\omega}_3(1 - c_\theta) + \hat{\omega}_2s_\theta \\ \hat{\omega}_1\hat{\omega}_2(1 - c_\theta) + \hat{\omega}_3s_\theta & c_\theta + \hat{\omega}_2^2(1 - c_\theta) & \hat{\omega}_2\hat{\omega}_3(1 - c_\theta) - \hat{\omega}_1s_\theta \\ \hat{\omega}_1\hat{\omega}_3(1 - c_\theta) - \hat{\omega}_2s_\theta & \hat{\omega}_2\hat{\omega}_3(1 - c_\theta) + \hat{\omega}_1s_\theta & c_\theta + \hat{\omega}_3^2(1 - c_\theta) \end{bmatrix} \end{aligned}$$

Matrix Logarithm Method

- If we take the trace of the matrix, we can solve for θ :
 - $tr(R) := r_{11} + r_{22} + r_{33} = 1 + 2 \cos \theta$

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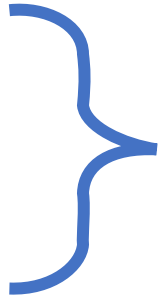
- $\text{tr}(R) := r_{11} + r_{22} + r_{33} = 1 + 2 \cos \theta$

- If we compute $R - R^\top$, we get:

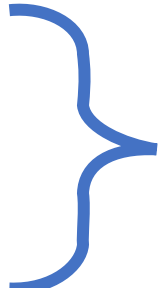
- $r_{32} - r_{23} = 2\hat{\omega}_1 \sin \theta$

- $r_{13} - r_{31} = 2\hat{\omega}_2 \sin \theta$

- $r_{21} - r_{12} = 2\hat{\omega}_3 \sin \theta$


$$[\hat{\omega}] = \frac{1}{2 \sin \theta} (R - R^\top)$$

Matrix Logarithm Method

- If we take the trace of the matrix, we can solve for θ :
 - $\text{tr}(R) := r_{11} + r_{22} + r_{33} = 1 + 2 \cos \theta$
- If we compute $R^\top - R$, we get:
 - $r_{32} - r_{23} = 2\hat{\omega}_1 \sin \theta$
 - $r_{13} - r_{31} = 2\hat{\omega}_2 \sin \theta$
 - $r_{21} - r_{12} = 2\hat{\omega}_3 \sin \theta$
$$[\hat{\omega}] = \frac{1}{2 \sin \theta} (R - R^\top)$$
- But what if $\sin \theta = 0$? (called singularities)
 - If $\theta = 2k\pi$, we have rotated by 360 degrees
 - If $\theta = (2k + 1)\pi$, then Rodrigues formula is $R = I + 2[\hat{\omega}]^2$, which gives:
$$\hat{\omega}_i = \pm \sqrt{\frac{r_{ii} + 1}{2}} \text{ and } 2\hat{\omega}_i \hat{\omega}_j = r_{ij} \text{ for } i \neq j$$

On to Homogeneous Transformations!

Homogeneous Transformations: $SE(3)$

The **special Euclidean group** $SE(3)$ is the set of 4×4 matrices of the form:

$$T = T(R, p) = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$$

Homogeneous Transformations: $SE(3)$

The **special Euclidean group** $SE(3)$ is the set of 4×4 matrices of the form:

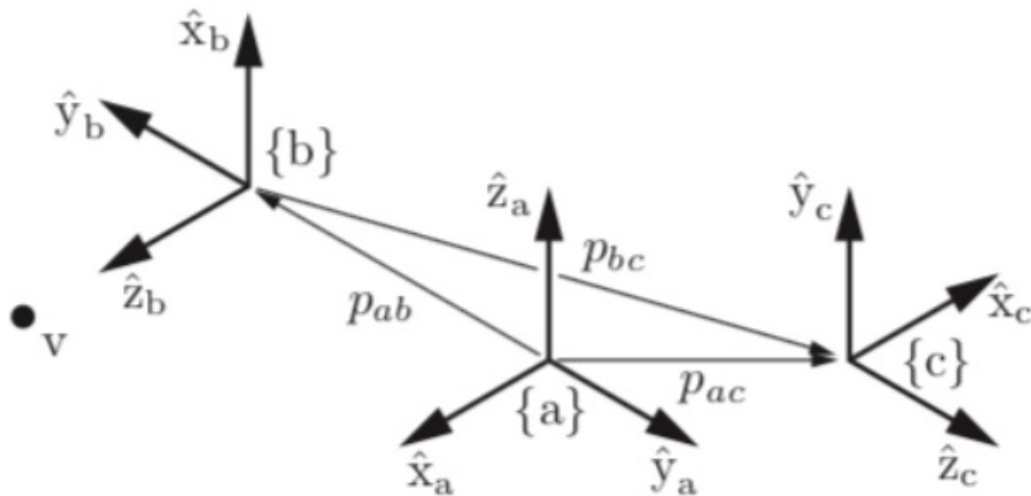
$$T = T(R, p) = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$$

- The inverse of T is $T^{-1} = \begin{bmatrix} R^\top & -R^\top p \\ 0 & 1 \end{bmatrix} \in SE(3)$
- If T_1 and $T_2 \in SE(3)$, then $T_1 T_2 \in SE(3)$
- T satisfies: $\|Tx - Ty\| = \|x - y\|$ and

$$(Tx - Tz)^\top (Ty - Tz) = (x - z)^\top (y - z)$$

Representing a configuration with SE(3)

Each frame can represent a body frame in a multi-link mechanism



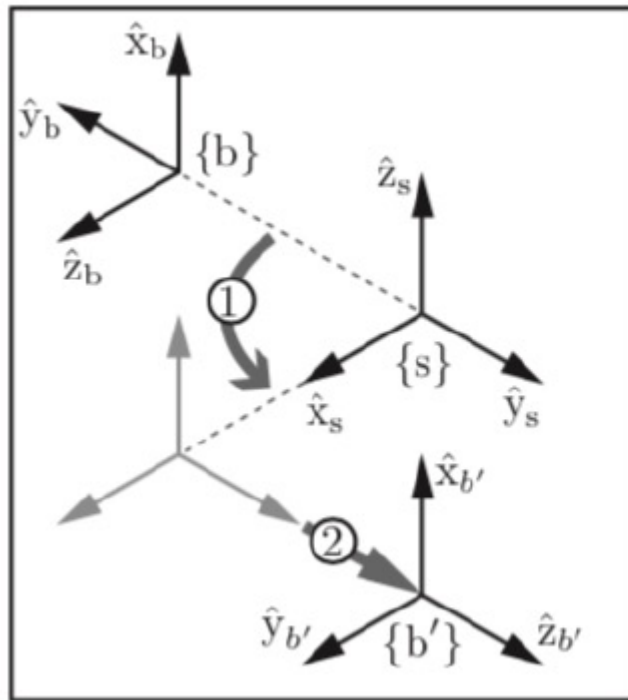
As before:

$$T_{ab}T_{bc} = T_{ac}$$

$$T_{ab}v_b = v_a$$

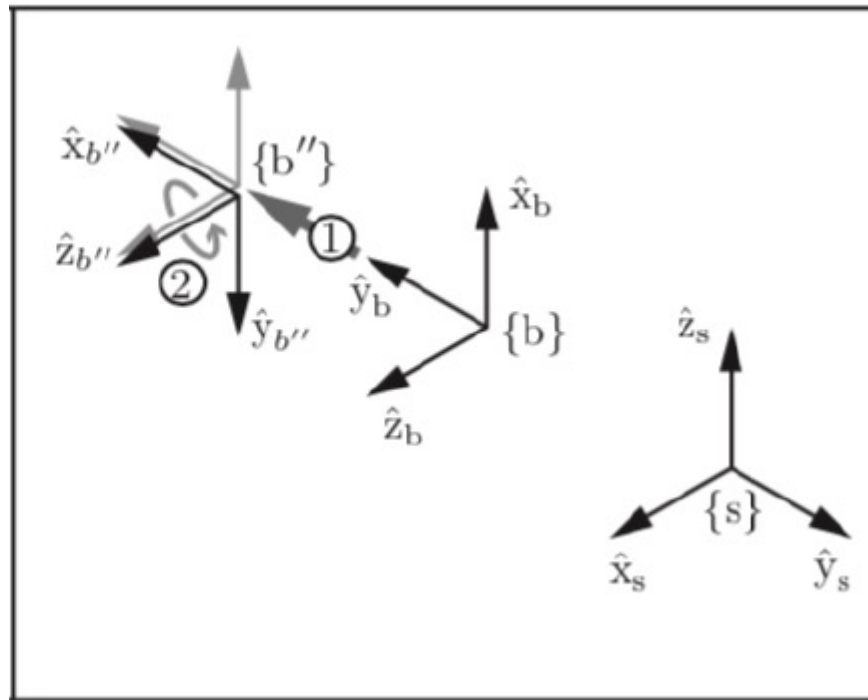
Displacing Frames (1)

Example Displacement (1)



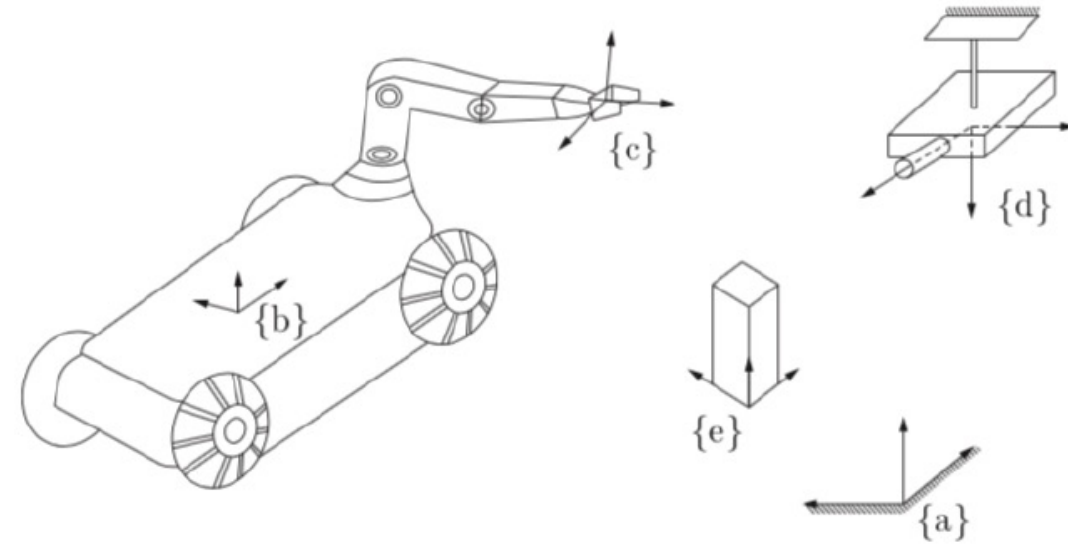
Displacing Frames (2)

Example Displacement (2)



Mobile arm example

- Robot arm mounted on wheeled platform. Camera fixed to ceiling.
- $\{b\}$ is body frame, $\{c\}$ end-effector frame, $\{e\}$ frame of object, and $\{a\}$ is fixed frame.
- We assume the camera position and orientation in $\{a\}$ is given.
- From camera measurements, you can evaluate the position and orientation of the body and the object in the camera frame.
- Since we designed our robot and have joint-angle estimates, we can obtain the end-effector position and orientation in the body frame.
- To pick up the object, we need the object position and orientation in the frame of our end-effector.



Summary

- Introduced **exponential coordinates** that allow us to parameterize rotations by the rotation axis $\hat{\omega}$ and the angle of rotation θ
- Walked through the **matrix logarithm** method, which tells us how to recover the exponential coordinates from a rotation matrix
- Started discussing **homogeneous transformations** where we not only look at rotations, but also translations in a single operation