Lecture 04: Rigid Body Motions

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Notes from Modern Robotics Ch 3.1-3.2

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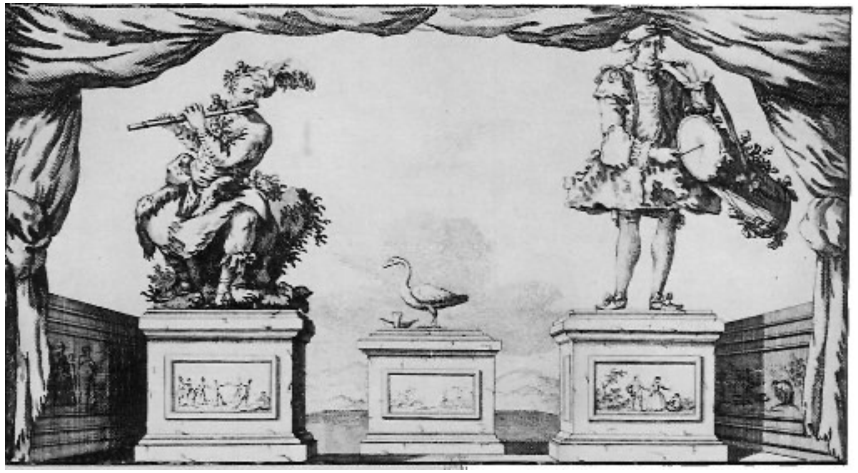
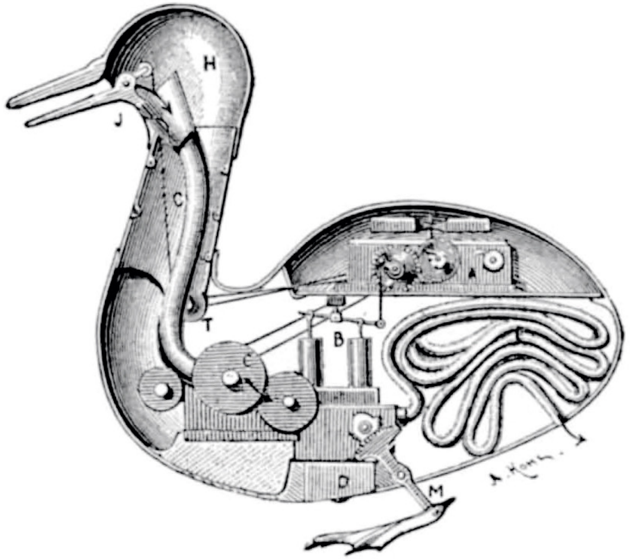
Vaucanson’s Duck

- On 30 May 1739 in France, Jacques de Vaucanson unveiled a mechanical duck
- The duck had over 400 moving parts in each wing alone, and could flap its wings, drink water, “digest” grain, and seemingly defecate
- Robert-Houdin described this as “a piece of artifice I would happily have incorporated in a conjuring trick”
- Voltaire wrote in 1741 that “Without the voice of le Maure and Vaucanson's duck, you would have nothing to remind you of the glory of France."





INTERIOR OF VAUCANSON’S AUTOMATIC DUCK.

A, clockwork; *B*, pump; *C*, mill for grinding grain; *F*, intestinal tube; *J*, bill; *H*, head; *M*, feet.

Credit: Wikipedia

Frames of Reference

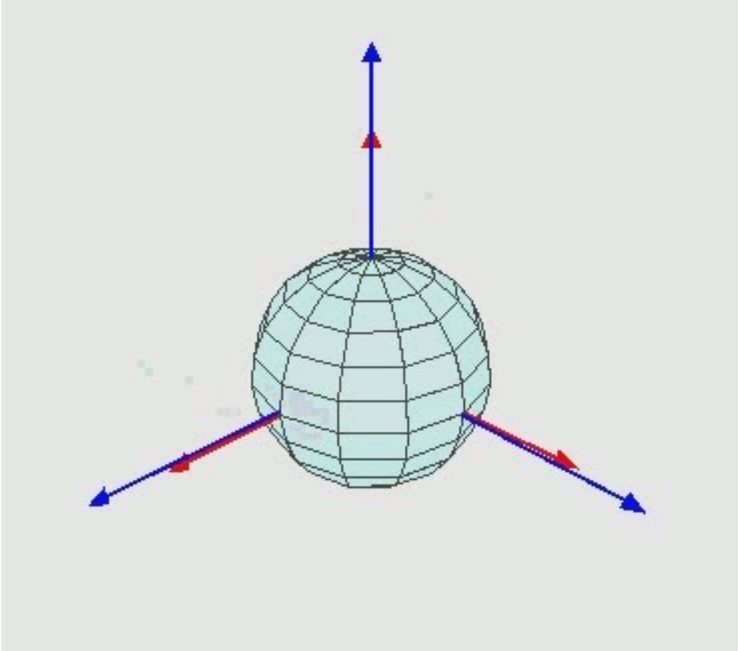
Rotation Matrices (1)

- 2D rotations:

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

- 3D rotations about a main axis:

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$



Credit: Wikipedia

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- If we rotate a vector v around the $\widehat{\omega}$ axis by angle θ, we say $\mathrm{Rot}(\widehat{\omega}, \theta)v$
  - For a rotation about the x-axis: $\mathrm{Rot}(\hat{x}, \theta)v = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} v$
  - Rotations about the y and z axis follow

Special Orthogonal Groups

$\boldsymbol{SO(3)}$, the group that represents all spatial rotation matrices, is the set of all 3×3 real matrices R that satisfy

1. $R^{\top}R = I$
2. $\det R = 1$

- Note that (2) implies that both (1) holds and that we obey the righthand rule

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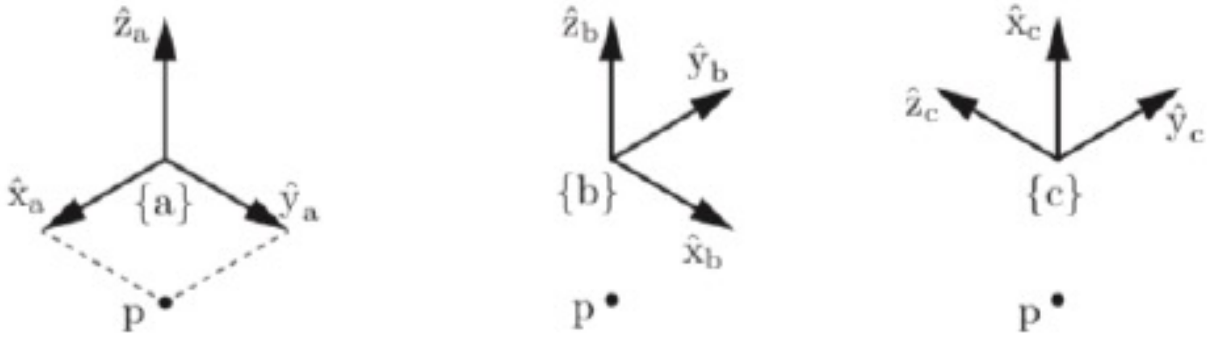
Similarly, $\boldsymbol{SO(2)}$, the group describes all planar rotation matrices, is the set of all 2×2 real matrices R, satisfying the same conditions.

Changing frames

Frames in 3D



$\hat{z}_a$
$\hat{x}_a$
$\{a\}$
$\hat{y}_a$
p
$\hat{z}_b$
$\hat{y}_b$
$\{b\}$
$\hat{x}_b$
p
$\hat{x}_c$
$\hat{z}_c$
$\hat{y}_c$
$\{c\}$
p

Affine Transformations (more next lecture!)

Rigid Body Motions

Recall the cross product

- The cross-product is defined as
$$a \times b = \|a\| \|b\| \sin\theta \, \hat{n}$$

- In coordinates:
$$a \times b = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

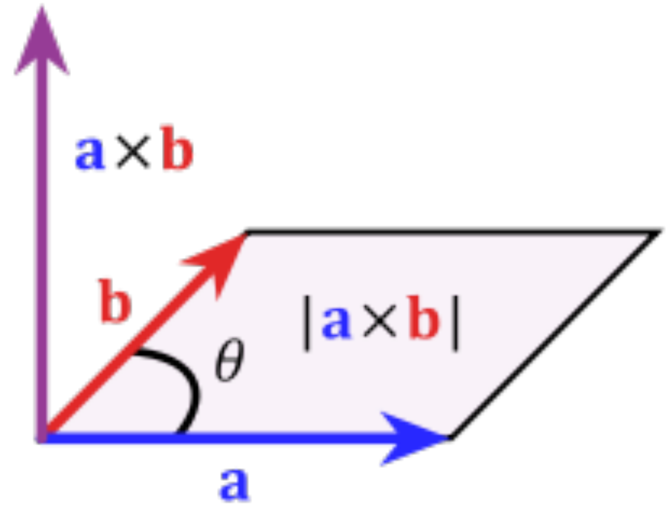
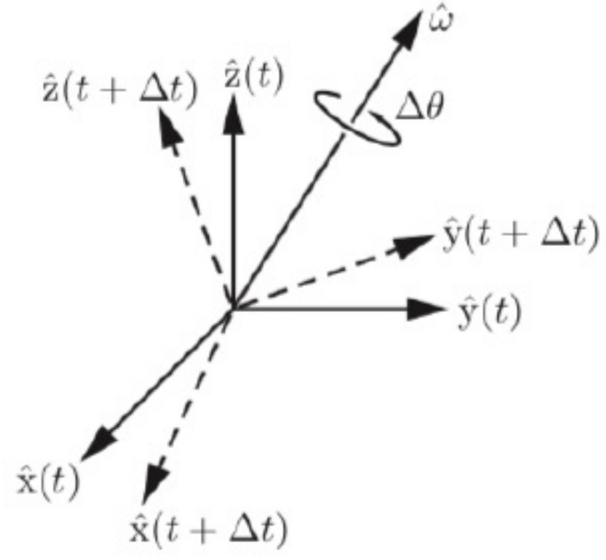
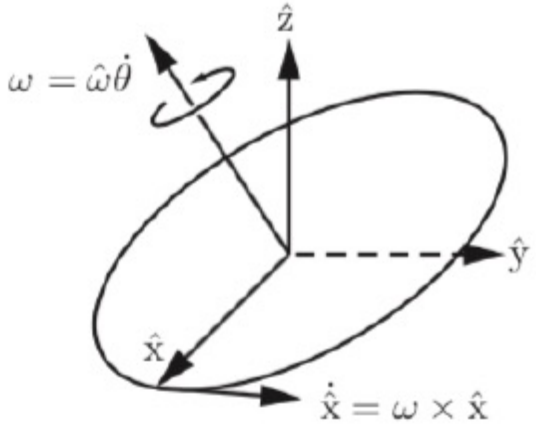


Image Credit: Wikipedia

Angular Velocities





Angular Velocities in Reference Frame

Some useful properties and relations

Given any $\omega \in \mathbb{R}^3$ and $R \in SO(3)$, the following holds: $R[\omega]R^\top = [R\omega]$

Now recall: $[\omega_s] = \dot{R}R^\top$

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If R is R_{sb}, we have that $\omega_s = R\omega_b \Leftrightarrow \omega_b = R^\top \omega_s$

$$[\omega_b] = [R^\top \omega_s] = R^\top \left(\dot{R}R^\top\right)R = R^\top \dot{R} = R^{-1}\dot{R}$$

This gives us: $[\omega_s] = \dot{R}R^\top$ and $[\omega_b] = R^\top \dot{R}$

Summary

- Formally defined **frames of reference** and **SO(3),** and built intuition for rigid body *motion*
- Defined **bracket notation** $[\boldsymbol{\omega}]$**,** a skew-symmetric representation of our angular velocity
- **Next time:** Affine transformation for more complex forms of motion!