Introduction to Robotics
Lecture 14: Lagrangian dynamics

Dynamics of open chains

- In the previous lectures, it was implicitly assumed that the robots' links had negligible mass , at least compared to the actuation power of their actuators. In this case, the kinematic approach describes motions well.
- We look now into the effects of non-negligible masses, and thus inertia, on the dynamics of robots.
- Inverse dynamics: determine the torques corresponding to a given state $(\theta, \dot{\theta})$:

$$\tau = M(\theta)\ddot{\theta} + h(\theta, \dot{\theta})$$

- Forward dynamics: determine the accelerations $\ddot{\theta}$ given $(\theta, \dot{\theta})$ and τ:

$$\ddot{\theta} = M^{-1}(\theta)(\tau - h(\theta, \dot{\theta}))$$

- $M(\theta)$ is called the *mass matrix*

Lagrangian dynamics

- We first review the Lagrangian approach to determine the dynamics of a rigid body.
- Denote by $q \in \mathbb{R}^n$ the so-called *generalized coordinates* of the system. These are a set of coordinates describing its state.
- From the generalized coordinates, we define the *generalized forces* $f \in \mathbb{R}^n$. These are the forces on the system as they "act" on the generalized coordinates.
- The pair needs to be consistently chosen so that the *power* dissipated by the system is $f^\top \dot{q}$.
- Denote by $K(q, \dot{q})$ the *kinetic* energy of the system, and by $P(q)$ its *potential energy*. (Recall that potential energy does *not* depend on $\dot{q}$). The *Lagrangian* of the system is defined as

$$L(q, \dot{q}) = K(q, \dot{q}) - P(q)$$

- From the Lagrangian of the system, we obtain the equations of motion through the principle of least action:

$$f = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q}.$$

Lagrangian dynamics: point mass

- Consider a point mass (particle) m constrained to move on the vertical line.
- A generalized coordinate is its height $x \in \mathbb{R}$.
- Suppose that an external force f is applied on it, and gravity is given by mg.
- The Lagrangian is

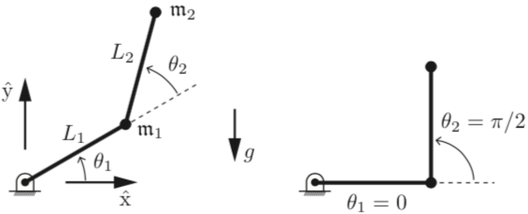
$$L(x, \dot{x}) = K(x, \dot{x}) - P(x) = \underbrace{\frac{1}{2}m\dot{x}^2}_{kin.\ en.} - \underbrace{mgx}_{pot.\ en.}.$$

- The equations of motion are given by

$$f = \frac{d}{dt}(m\dot{x}) - (-mg) = m\ddot{x} + mg.$$

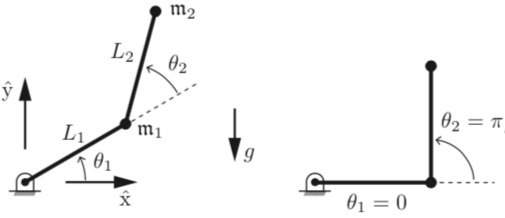
- We obtain the same equations of motion using Newton's $f = ma$ law.

Lagrangian dynamics: 2R open chain



- Consider a 2R open chain, with links of masses m_1, m_2 respectively. To simplify things, we assume that the masses are concentrated at the ends of links.
- We take the joint positions (θ_1, θ_2) for generalized coordinates, and (τ_1, τ_2), the torques applied at the joints, as generalized forces. Note that $\tau^\top \dot{\theta}$ is the power dissipated by the torques.
- We now need to derive the Lagrangian. To this end, we need the position and velocity of the masses.

Lagrangian dynamics: 2R open chain



$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} L_1 \cos\theta_1 \\ L_1 \sin\theta_1 \end{bmatrix} \text{ and } \begin{bmatrix} \dot{x}_1 \\ \dot{y}_1 \end{bmatrix} = \begin{bmatrix} -L_1 \sin\theta_1 \\ L_1 \cos\theta_1 \end{bmatrix} \dot{\theta}_1.$$

For link 2:

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) \\ L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2) \end{bmatrix} \text{ and}$$

$$\begin{bmatrix} \dot{x}_2 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} -L_1 \sin\theta_1 - L_2 \sin(\theta_1 + \theta_2) & -L_2 \sin(\theta_1 + \theta_2) \\ L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) & L_2 \cos(\theta_1 + \theta_2) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}.$$

Lagrangian dynamics: 2R open chain

- Using the relations of the previous slide, we obtain the kinetic energy of the links:

$$
\begin{aligned}
K_1 &= \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) = \frac{1}{2}m_1 L_1^2\dot{\theta}_1^2 \\
K_2 &= \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) \\
&= \frac{1}{2}m_2\left((L_2^2 + 2L_1L_2\cos\theta_2 + L_2^2)\dot{\theta}_1^2 + 2(L_2^2 + L_1L_2\cos\theta_2)\dot{\theta}_1\dot{\theta}_2 \right. \\
&\left. + L_2^2\dot{\theta}_2^2\right)
\end{aligned}
$$

- The potential energies of the links are

$$
\begin{aligned}
P_1 &= m_1 g y_1 = mgL_1\sin\theta_1 \\
P_2 &= m_2 g y_2 = m_2 g(L_1\sin\theta_1 + L_2\sin(\theta_1+\theta_2))
\end{aligned}
$$

Lagrangian dynamics: 2R open chain

- The equations of motion are $\tau_i = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_i} - \frac{\partial L}{\partial \theta_1}$, $i = 1, 2$. This yields here

$$
\begin{aligned}
\tau_1 &= \left(\mathfrak{m}_1 L_1^2 + \mathfrak{m}_2(L_1^2 + 2L_1L_2\cos\theta_2 + L_2^2)\right)\ddot{\theta}_1 \\
&\quad + \mathfrak{m}_2(L_1L_2\cos\theta_2 + L_2^2)\ddot{\theta}_2 - \mathfrak{m}_2L_1L_2\sin\theta_2(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2) \\
&\quad + (\mathfrak{m}_1 + \mathfrak{m}_2)L_1 g\cos\theta_1 + \mathfrak{m}_2 g L_2\cos(\theta_1 + \theta_2), \\
\tau_2 &= \mathfrak{m}_2(L_1L_2\cos\theta_2 + L_2^2)\ddot{\theta}_1 + \mathfrak{m}_2L_2^2\ddot{\theta}_2 + \mathfrak{m}_2L_1L_2\dot{\theta}_1^2\sin\theta_2 \\
&\quad + \mathfrak{m}_2 g L_2\cos(\theta_1 + \theta_2).
\end{aligned}
$$

- We can write the above equations as

$$
\tau = M(\theta)\ddot{\theta} + \underbrace{c(\theta, \dot{\theta}) + g(\theta)}_{h(\theta,\dot{\theta})}
$$

with the definitions

$$
M(\theta) = \begin{bmatrix} \mathfrak{m}_1L_1^2 + \mathfrak{m}_2(L_1^2 + 2L_1L_2\cos\theta_2 + L_2^2) & \mathfrak{m}_2(L_1L_2\cos\theta_2 + L_2^2) \\ \mathfrak{m}_2(L_1L_2\cos\theta_2 + L_2^2) & \mathfrak{m}_2L_2^2 \end{bmatrix}
$$

$$
c(\theta, \dot{\theta}) = \begin{bmatrix} -\mathfrak{m}_2L_1L_2\sin\theta_2(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2) \\ \mathfrak{m}_2L_1L_2\dot{\theta}_1^2\sin\theta_2 \end{bmatrix},
$$

$$
g(\theta) = \begin{bmatrix} (\mathfrak{m}_1 + \mathfrak{m}_2)L_1 g\cos\theta_1 + \mathfrak{m}_2 g L_2\cos(\theta_1 + \theta_2) \\ \mathfrak{m}_2 g L_2\cos(\theta_1 + \theta_2) \end{bmatrix},
$$

Lagrangian dynamics: 2R open chain

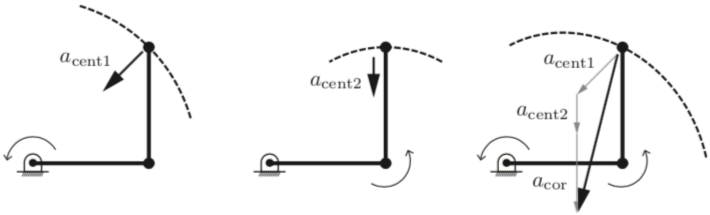
- The matrix $M(\theta)$ is symmetric and positive definite. It is called the mass matrix.
- The vector $c(\theta, \dot{\theta})$ contains the centripetal and Coriolis forces/torques, and $g(\theta)$ contains the gravitational forces/torques.
- We could have obtained the same equations again from $f = ma$

$$f_1 = \begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{z1} \end{bmatrix} = \mathfrak{m}_1 \begin{bmatrix} \ddot{x}_1 \\ \ddot{y}_1 \\ \ddot{z}_1 \end{bmatrix} = \mathfrak{m}_1 \begin{bmatrix} -L_1\dot{\theta}_1^2 c_1 - L_1\ddot{\theta}_1 s_1 \\ -L_1\dot{\theta}_1^2 s_1 + L_1\ddot{\theta}_1 c_1 \\ 0 \end{bmatrix},$$

$$f_2 = \mathfrak{m}_2 \begin{bmatrix} -L_1\dot{\theta}_1^2 c_1 - L_2(\dot{\theta}_1 + \dot{\theta}_2)^2 c_{12} - L_1\ddot{\theta}_1 s_1 - L_2(\ddot{\theta}_1 + \ddot{\theta}_2) s_{12} \\ -L_1\dot{\theta}_1^2 s_1 - L_2(\dot{\theta}_1 + \dot{\theta}_2)^2 s_{12} + L_1\ddot{\theta}_1 c_1 + L_2(\ddot{\theta}_1 + \ddot{\theta}_2) c_{12} \\ 0 \end{bmatrix}$$

- Note that since $(\hat{x}, \hat{y})$ is an inertial frame, we have equations $\ddot{x} = \cdots, \ddot{y} = \cdots$. The frame $(\hat{\theta}_1, \hat{\theta}_2)$ is not inertial, hence there is a non-trivial M in front of $\ddot{\theta}$.

Lagrangian dynamics: 2R open chain

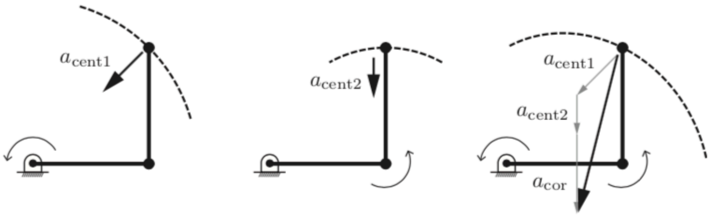


- A zero acceleration in a non-inertial frame does not imply a zero acceleration in an inertial frame.
- Consider the arm in position $(\theta_1, \theta_2) = (0, \pi/2)$. Assuming $\ddot{\theta} = 0$, we have

$$\begin{bmatrix} \ddot{x}_2 \\ \ddot{y}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -L_1\dot{\theta}_1^2 \\ -L_2\theta_1^2 - L_2\dot{\theta}_2^2 \end{bmatrix}}_{\text{centripetal}} + \underbrace{\begin{bmatrix} 0 \\ -2L_2\dot{\theta}_1\dot{\theta}_2 \end{bmatrix}}_{\text{Coriolis}}$$

- Quadratic terms $\dot{\theta}_i^2$ are called centripetal terms, the mixed quadratic terms $\dot{\theta}_1\dot{\theta}_2$ the Coriolis terms.

Lagrangian dynamics: 2R open chain



- If $\dot{\theta}_2 = 0$, no Coriolis and centrifugal accel. is $(-L_1\dot{\theta}_1^2, -L_2\dot{\theta}_1^2)$. Similarly, $\dot{\theta}_1 = 0$, no Coriolis and centrifugal accel. is $(0, -L_2\dot{\theta}_2^2)$. These accelerations keep the mass rotating around the joints 1 and 2 respectively.
- The Coriolis force appears if both $\dot{\theta}_i$ are non-zero. Note that its sign depends on the signs of the θ_i's.

Lagrangian dynamics for general chains

- For a general open chain with n links, we take the link angles θ_i as generalized coordinates and the corresponding torques τ_i as generalized forces.
- The kinetic energy can always be written as

$$K(\theta, \dot{\theta}) = \dot{\theta}^\top M(\theta) \dot{\theta}$$

  for an appropriately defined mass matrix $M(\theta)$.
- The dynamics equation are then

$$\tau_i = \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}_i} - \frac{\partial L}{\partial \theta_i}$$

  for $L = K - P$ with $P(\theta)$ the potential energy of the system.

Lagrangian dynamics for general chains

- Written explicitly, we have

$$\tau_i = \sum_{j=1}^{n} m_{ij}(\theta)\ddot{\theta}_j + \sum_{j,k=1}^{n} \Gamma_{ijk}(\theta)\dot{\theta}_j\dot{\theta}_k + \frac{\partial P}{\partial \theta_i}$$

where

$$\Gamma_{ijk}(\theta) = \frac{1}{2}\left(\frac{\partial m_{ij}}{\partial \theta_k} + \frac{\partial m_{ik}}{\partial \theta_j} - \frac{\partial m_{jk}}{\partial \theta_i}\right)$$

are the Christoffel's symbols of the first kind.

- This dynamics is also written as

$$\tau = M(\theta)\ddot{\theta} + C(\theta,\dot{\theta})\dot{\theta} + g(\theta),$$

where M is the mass matrix, and $C(\theta,\dot{\theta}) \in \mathbb{R}^{n\times n}$ is the matrix with entries

$$c_{ij} = \sum_{k=1}^{n} \Gamma_{ijk}(\theta)\dot{\theta}_k.$$

It is called the Coriolis matrix.