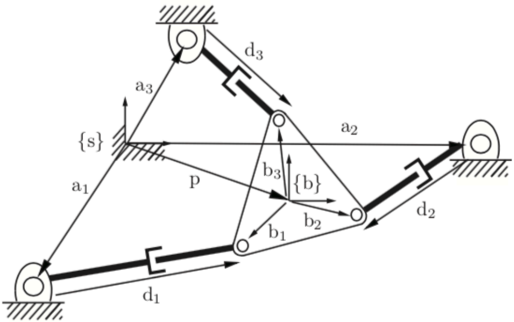
Introduction to Robotics
Lecture 13: Kinematics of closed chains

Kinematics of closed chains

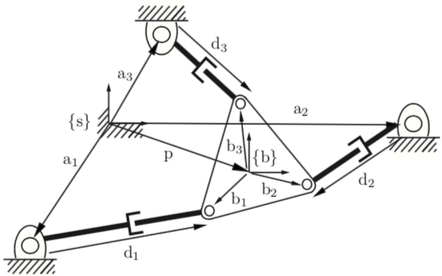
- A kinematic chain that contains one or more loops (series of links joining the ground to the ground, i.e. without a free end-effector) is called a closed chain.
- Unlike open chains, closed-chain can have non-actuated, or passive, joints.
- Kinematics of closed-chains is more complex than the one of open chains, since: 1) there are algebraic equations corresponding to loops in mechanism. These may be independent or dependent depending on mechanism. 2) Some joint are not actuated. 3) Very often redundant.
- Approach: write algebraic equations corresponding to loops in the mechanism.

Kinematics of closed chains



- 3 DOFs planar 3× RPR mechanism.
- The three prismatic joints are actuated, the 6 revolute joints are passive.
- Denote length of the 3 legs by s_1, s_2, s_3 ($s_i = \|d_i\|$) and by T_{sb} the orientation of the body frame.
- Forward kinematics: $(s_1, s_2, s_3) \mapsto T_{sb}$
- Inverse kinematics: $T_{sb} \mapsto (s_1, s_2, s_3)$.

Kinematics of closed chains: 3RPR mechanism



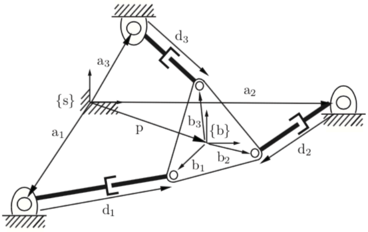
- Closed loops in the mechanism yield:

$$d_i = p + b_i - a_i.$$

We set $a_i = (a_{ix}, a_{iy})$ in s-frame coordinates, and similarly for p, d, and $b_i = (b_{ix}, b_{iy})$ in b-frame coordinates. If we denote by R_{sb} the rotation matrix of T_{sb}, we have

$$\begin{bmatrix} d_{ix} \\ d_{iy} \end{bmatrix} = \begin{bmatrix} p_x \\ p_y \end{bmatrix} + R_{sb} \begin{bmatrix} b_{ix} \\ b_{iy} \end{bmatrix} - \begin{bmatrix} a_{ix} \\ a_{iy} \end{bmatrix}$$

Kinematics of closed chains: 3RPR mechanism



- We have

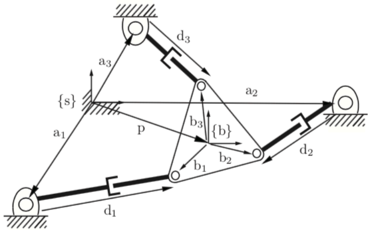
$$\begin{bmatrix} d_{ix} \\ d_{iy} \end{bmatrix} = \begin{bmatrix} p_x \\ p_y \end{bmatrix} + R_{sb} \begin{bmatrix} b_{ix} \\ b_{iy} \end{bmatrix} - \begin{bmatrix} a_{ix} \\ a_{iy} \end{bmatrix}$$

  and $s_i^2 = d_{ix}^2 + d_{iy}^2$, which yields

$$s_i^2 = (p_x + b_{ix} \cos\phi - b_{iy} \sin\phi - a_{ix})^2 + (p_y + b_{ix} \sin\phi + b_{iy} \cos\phi - a_{iy})^2,$$

  where R_{sb} is a rotation matrix by angle ϕ.
- The above equation solves the *inverse* kinematics problem: given T_{sb}, and thus ϕ, p, we can get the s_i.

Kinematics of closed chains: 3RPR mechanism

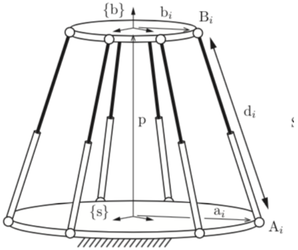


- Forward kinematics is harder. One needs to solve the equations above for p, ϕ numerically in general.
- Using the substitutions

$$t = \tan\frac{\phi}{2}, \sin\phi = \frac{2t}{1+t^2}, \cos\phi = \frac{1-t^2}{1+t^2},$$

  we can reduce the three equations to a polynomial equation in t of degree 6.

Closed chains: 6SPS Steward-Gough platform



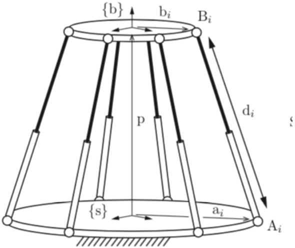
- The 12 spherical joints (at the ends of the 6 arms) are passive, the 6 prismatic joints (middle of the arms) are actuated.
- We make the definitions: p, a_i, d_i are expressed in s-frame, b_i in b-frame, and $R = R_{sb} \in SO(3)$.
- We have the 6 equations

$$d_i = p + Rb_i - a_i.$$

Denote by s_i the length of leg i. We have

$$s_i^2 = d_i^\top d_i = (p + Rb_i - a_i)^\top (p + Rb_i - a_i)$$

Closed chains: 6SPS Steward-Gough platform



- We have

$$s_i^2 = d_i^\top d_i = (p + Rb_i - a_i)^\top (p + Rb_i - a_i).$$

  From this equation, the inverse kinematics is straightforward: given R, p, we can obtain s_i.
- The forward kinematics requires numerically solving the 6 equations above.

Velocity kinematics of closed chains

- Velocity kinematics for closed chains is generally difficult, and no nice systematic formulas (like PoE) can be used to obtain the Jacobian in general.
- The velocity kinematics is then obtained from first principles, by differentiating the forward or inverse kinematics map to obtain a Jacobian relating actuator velocities to the body twist.
- For example, the two previous examples produced analytic inverse kinematics maps: $s = g(R, p)$. We can differentiate these maps to obtain

$$\dot{s} = \frac{\partial g}{\partial(R, p)} \mathcal{V}_s,$$

  where $\frac{\partial g}{\partial(\omega,\theta,p)}$ is a matrix of partial derivatives and we expressed $R = Rot(\omega, \theta)$. This requires very lengthy computations.

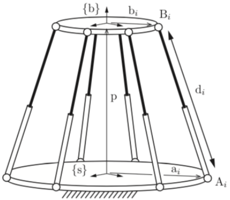
Velocity kinematics of closed chains

- We can however obtain relatively easily the inverse Jacobian from a static analysis.
- Recall that the Jacobian relates a wrench to the actuator forces according to

$$\mathcal{F}_s^\top J_s = \tau^\top.$$

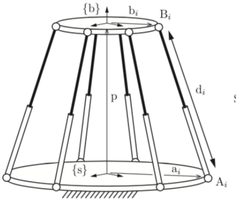
  If we have a 6DOFs mechanism and 6 actuators, the Jacobian is square and can be inverted at non-singular configurations.
- A static analysis of the mechanism can yield J^{-T}.

Closed chains: 6SPS Steward-Gough platform



- If no external forces, the only forces on the platform are at the spherical joints. Write these forces as $f_i = \tau_i \hat{n}_i$ in the s-frame.
- The moment the forces generate is $m_1 = r_i \times f_i$, where r_i the vector joining the origin of the s-frame to the spherical joint i, expressed in s-frame.
- Since the spherical joints are passive, the force is aligned with the joint axis, and thus we can express the torque as $m_i = q_i \times f_i$, where q_i is the position of the lower spherical joint on the leg.

Closed chains: 6SPS Steward-Gough platform



- We thus have that the wrench on the moving platform is

$$\mathcal{F}_s = \sum_{i=1}^{6} \mathcal{F}_i = \sum_{i=1}^{6} \begin{bmatrix} r_\times \hat{n}_i \\ \hat{n}_i \end{bmatrix} \tau_i.$$

We can write it as

$$\mathcal{F}_s = \begin{bmatrix} -\hat{n}_1 \times q_i & \cdots & -\hat{n}_6 \times q_6 \\ \hat{n}_1 & \cdots & \hat{n}_6 \end{bmatrix} \begin{bmatrix} \tau_1 \\ \vdots \\ \tau_6 \end{bmatrix}$$

from which we obtain J_s^{-T}.