Introduction to Robotics
Lecture 15: Trajectory generation

Trajectories and paths

- The specification of the robot's state as a function of time is called a *trajectory*.
- Using forward kinematics maps, we can obtain the position of each link from the knowledge of the joint angles.
- We can consider a trajectory to be given by

$$\theta : [0, T] \rightarrow \mathbb{R}^n,$$

  a function from a time interval to $\mathbb{R}^n$. $\theta(t)$ is the value of the joint angles at time t.
- The trajectory of the end-effector is then $T_{sb}(\theta(t))$

Trajectories and paths

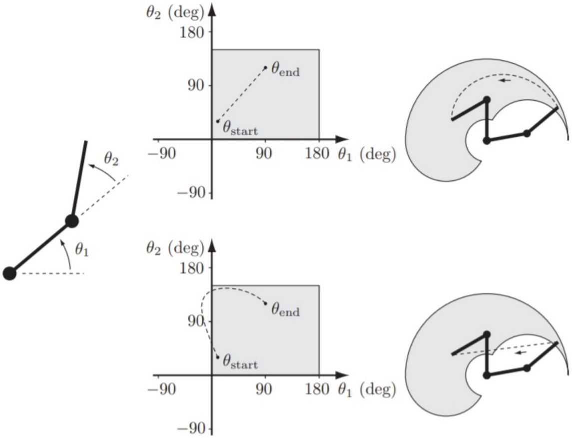
- A *path* is a continuous set of points (state). It is a purely geometric object and does not make any reference to time.
- For example, the centered unit circle in $\mathbb{R}^2$ is a path, or a line segment joining $(0,0)$ to $(1,1)$ is a path.
- A path can describe the set of states we want the robot to follow, but without allusion to the time at which we want the robot to be at a particular state.
- A path + specification of time (or time-scaling) yields a trajectory.
- The image of a trajectory, i.e. $\text{im}(\theta)$ is a path.
- For example, $\theta_1(t) = (t, t)$ and $\theta_2(t) = (t^2, t^2)$, both for $t \in [0,1]$ are two trajectories that *trace* the same path.
- Similarly, $\theta_1(t) = (\cos 2\pi t, \cos 2\pi t), t \in [0,1]$ and $\theta_2(t) = (\cos \pi t, \cos \pi t), t \in [0,2]$ trace the same paths.

Trajectories and paths

- In the second example of the previous slide, the trajectory covers the path twice as fast, and in the first example, they take the same amount of time but θ_2 starts slow and ends fast.
- By convention, we describe a path as the image of a normalized trajectory $\theta(s)$, where $s \in [0, 1]$.
- A time scaling $s(t)$ is a monotonically increasing function $s : [0, T] \to [0, 1]$. A trajectory tracing θ can then be written as

$$\theta_1(t) = \theta(s(t))$$

Straight-line in joint space is not straight in workspace



Straight-line paths

- A straight-line in joint space from an initial configuration θ_0 to an end-configuration θ_1 is given by the path
$$\theta(s) = \theta_0 + s(\theta_1 - \theta_0)$$
- Now assume that the end-effector space in $\mathbb{R}^2$ or $\mathbb{R}^3$, i.e. the orientation of the end-effector is ignored. A straight-line in joint-space will generally not correspond to a straight-line in end-effector space.
- For example, if $X_0 = T(\theta_0)$ and $X_1 = T(\theta_1)$ with T the end-effector kinematics map. Then $X(\theta_1(t))$ is not a straight-line.
- We can specify a straight-line in end-effector space as

$$X(s) = X_0 + s(X_1 - X_0).$$

Straight-line paths

- If the end-effector space is $SE(3)$, note that for $X_0, X_1 \in SE(3)$, then $X(s) = X_0 + s(X_1 - X_0)$ is *not* an element of $SE(3)$ for $s \in [0, 1]$.
  How to think of a straight-line in $SE(3)$?
- A straight-line in $\mathbb{R}^2$ is characterized by a constant velocity. Indeed, the solution of $\dot{x} = v$, with v constant is $x(t) = x_0 + vt$. We adopt this as a definition for straight-lines in $SE(3)$.
- Recall that $\dot{T} = T[\mathcal{S}]$, and if $\mathcal{S}$ is constant, then $T(t) = T_0 e^{[\mathcal{S}]t}$, with $T_0 \in SE(3)$. We use this equation and say that if $\mathcal{S}$ is constant, then $T(t)$ as above is a straight-line in $SE(3)$.
- Given X_0 and X_1, the straight-line in $SE(3)$ joining X_0 to X_1 is

$$X(s) = X_0 e^{\log(X_0^{-1} X_1)s}.$$

Straight-line paths

- If we set $T(t) = (R(t), p(t))$ for the $T(t)$ defined above, we obtain

$$p(s) = p_0 + s(p_1 - p_0) \text{ and } R(s) = R_0 e^{\log(R_0^\top R_1)s},$$

  which justifies calling such trajectory a straight-line.
- Using the inverse kinematics map, we obtain a path in joint space that translates into a straight-line path in $SE(3)$.

Time-scaling of straight-line paths

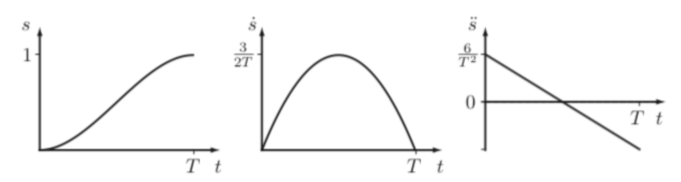
- A time-scaling s of a path is used to insure that the motion is smooth, and that constraints on the maximal velocity and acceleration or the robot are met.
- Given a time-scaling $s(t)$, we have $\theta(t) = \theta_0 + s(t)(\theta_1 - \theta_0)$ and thus

$$\frac{d}{dt}\theta(t) = \frac{ds}{dt}(\theta_1 - \theta_0) \text{ and } \ddot{\theta}_t = \frac{d^2 s}{dt^2}(\theta_1 - \theta_0)$$

- We now want to choose a time-scaling so as to insure that $\dot{\theta}$ and $\ddot{\theta}$ are not too high.
- One option is to use a *parametric* form for $s(t)$. A popular choice is to use polynomials.

Polynomial time-scaling of straight-line paths

- Let
$$s(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3.$$
We need $s(0) = 0, s(T) = 1, \dot{s}(0) = 0$ and $\dot{s}(T) = 0$, specifying that we start from θ_0 with zero velocity, and reach θ_1 with zero velocity.
- We see that $\dot{s} = a_1 + 2a_2 t^3 + a_3 t^2$. The four constraints above yield
$$a_0 = 0, a_1 = 0, a_2 = \frac{3}{T^2} \text{ and } a_3 = -\frac{2}{T^3}.$$



Polynomial time-scaling of straight-line paths

- The above parameters yield the trajectory

$$\theta(t) = \theta_0 + \left(\frac{3t^2}{T^2} - \frac{2t^3}{T^3}\right)(\theta_1 - \theta_0)$$

with derivative and accelerations

$$\dot{\theta} = \left(\frac{6t}{T^2} - \frac{6t^2}{T^3}\right)(\theta_1 - \theta_0) \text{ and } \ddot{\theta} = \left(\frac{6}{T^2} - \frac{12t}{T^3}\right)(\theta_1 - \theta_0)$$

- The maximum velocity is obtained at $t = T/2$ and is $\dot{\theta}_{max} = \frac{3}{2T}(\theta_1 - \theta_0)$.
- The maximal joint accelerations are obtained at $t = 0$ and $t = T$ and are

$$\ddot{\theta}_{max} = \left|\frac{6}{T^2}(\theta_1 - \theta_0)\right| \text{ and } \ddot{\theta}_{min} = -\left|\frac{6}{T^2}(\theta_1 - \theta_0)\right|$$

- If we have limitations on maximal velocities and accelerations, we choose T appropriately.

Polynomial time-scaling of straight-line paths

- We choose a parametrization $s(t)$, and computed the resulting $\dot{\theta}$ and $\ddot{\theta}$ using this parametrization. We then found their maximal values and had one parameter, T, we could tune to meet requirements.
- We can follow the same procedure with different parametrizations for $s(t)$: e.g. polynomials of order 5, trapezoidal functions, splines, etc. The principles are the same.
- Having more parameters allows us to meet more constraints. For example, using a fifth order polynomial, we can ensure that $\ddot{\theta}(0) = \ddot{\theta}(T) = 0$. No "jerk" at the beginning and end of the motion.