Introduction to Robotics
Lecture 10. Velocity Kinematics: The Jacobian

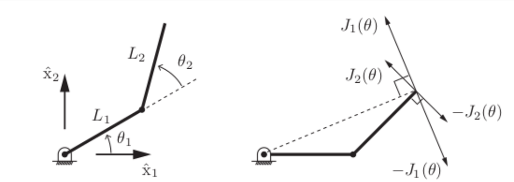
Velocity kinematics

- We know how to calculate the position of the end-effector of an open chain given the joint angles, i.e. derive the forward kinematic map.
- We now seek to evaluate the *twist*, i.e. the velocity, of the end-effector frame given the joint angles $(\theta_1, \ldots, \theta_n)$ and their velocities $(\dot{\theta}_1, \ldots, \dot{\theta}_n)$
- Abstractly, we can set the vector $x(t)$ to be the position of the end-effector at time t. The forward kinematics map is $x(t) = f(\theta(t))$. We want to obtain the derivative of $x(t)$:

  $$\frac{d}{dt}x(t) = \frac{\partial f}{\partial \theta}|_{\theta(t)}\dot{\theta}$$

  The matrix $\frac{\partial f}{\partial \theta}$ is called the *Jacobian* of f.
- We can think of the Jacobian as encoding the *sensitivity* of the motion of the end-effector with regard to motions of the joints.

Velocity kinematics: basic example



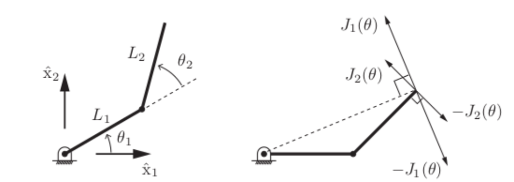
- The forward kinematics of this open chain is

$$x_1 = L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2)$$
$$x_2 = L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2)$$

- Now assume that $\theta_i = \theta_i(t)$ and differentiate on both sides

$$\dot{x}_1 = -L_1\dot{\theta}_1 \sin\theta_1 - L_2(\dot{\theta}_1 + \dot{\theta}_2) \sin(\theta_1 + \theta_2)$$
$$\dot{x}_2 = L_1\dot{\theta}_1 \cos\theta_1 + L_2(\dot{\theta}_1 + \dot{\theta}_2) \cos(\theta_1 + \theta_2)$$

Velocity kinematics: basic example



- We can rearrange the previous equation as follows

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -L_1 \sin\theta_1 - L_2 \sin(\theta_1 + \theta_2) & -L_2 \sin(\theta_1 + \theta_2) \\ L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) & L_2 \cos(\theta_1 + \theta_2) \end{bmatrix}}_{J(\theta)} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

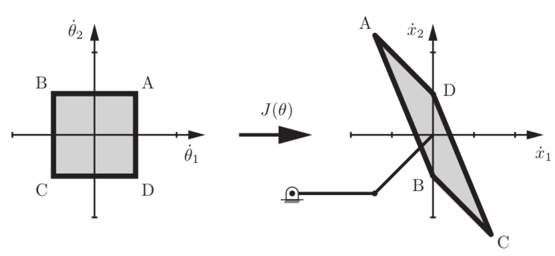
- We denote the two columns of $J(\theta)$ as $J_1(\theta)$ and $J_2(\theta)$ and get

$$\dot{x} = J_1(\theta)\dot{\theta}_1 + J_2(\theta)\dot{\theta}_2$$

Velocity kinematics: basic example

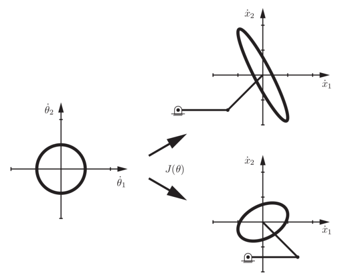
- In the equation $\dot{x} = J_1(\theta)\dot{\theta}_1 + J_2(\theta)\dot{\theta}_2$, we think of $\dot{\theta}_1$ and $\dot{\theta}_2$ as the coefficients of a linear combination of the vectors $J_1(\theta)$ and $J_2(\theta)$.
- If $J_1(\theta)$ and $J_2(\theta)$ are *linearly independent*, we can find coefficients $\dot{\theta}_i$ so that $\dot{x}$ takes on any value.
- Practically, this says that by choosing appropriate velocities for the joints, we can make the end-effector move in any desired directions.
- Note that the vectors are functions of θ. The values of θ for which the $J_i(\theta)$ are not linearly independent, or equivalently, $\det J(\theta) = 0$, are called *singular configurations*.
- At singular configurations, some directions of motions for the end-effector *cannot be realized*.
- For the example here, if $\theta_2 = 0$, then the configuration is singular.

The Jacobian and its uses



- Let us look at how velocities for θ_i are mapped to velocities for x.
- Set $L_1 = L_2 = 1$ and $\theta_1 = 0, \theta_2 = \pi/4$. We calculate $J(0, \pi/4) = \begin{bmatrix} -.71 & -.71 \\ 1.71 & .71 \end{bmatrix}$.
- Assume that the actuators allow joint velocities $\dot{\theta}_i \in [-1, 1]$. The set of possible joint velocities is mapped into a set of possible end-effector velocities by taking $\dot{x} = J(\theta)\dot{\theta}$.

The Jacobian and its uses



- If the joint velocities are so that $\dot{\theta}_1^2 + \dot{\theta}_2^2 \leq 1$ (disk of radius 1), we can map them through the Jacobian as before, and obtain the set of possible end-effector velocities.
- The ellipsoid obtained as an image of the disk in joint-velocities space is called the *manipulability* ellipsoid.
- A flatter ellipsoid says that we are close to a singular configuration: some directions are not available; a large joint velocity yields a small end-effector velocity.

The Jacobian and its uses

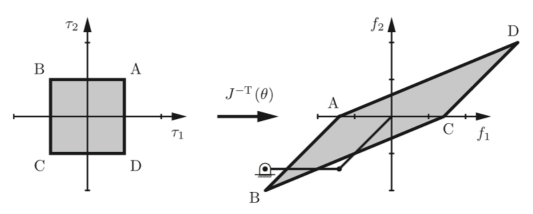
- Assume a force f_t is applied on the end-effector (e.g., weight of a load). What torque to apply at the joint to keep the end-effector at a fixed position?
- Let $\tau = (\tau_1, \tau_2)$ be the joints' torque vector. By conservation of power (principle of virtual work), we need

$$f_t^\top \dot{x} = \tau^\top \dot{\theta} \Rightarrow f_t^\top J(\theta)\dot{\theta} = \tau^\top \dot{\theta}$$

  for all $\dot{\theta}$. We conclude that

$$\tau = J^\top(\theta) f_t.$$

The Jacobian and its uses



- Reciprocally, given limits on the possible torques at the joints (and assuming that $J^\top$ is invertible!), we can plot all the forces that can be counteracted at the end effector as

$$f_t = (J^\top(\theta))^{-1}\tau$$

Computing the Jacobian from the FKM in PoE form

- Assume given the forward kinematics map in a product of exponential form

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} \cdots e^{[\mathcal{S}_n]\theta_n} M.$$

- Differentiating, we obtain

$$\begin{aligned}\dot{T} &= \frac{d}{dt}\left(e^{[\mathcal{S}_1]\theta_1}\right) e^{[\mathcal{S}_2]\theta_2} \cdots e^{[\mathcal{S}_n]\theta_n} M + \cdots \\ &\quad + e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} \cdots \frac{d}{dt}\left(e^{[\mathcal{S}_n]\theta_n}\right) M \\ &= [\mathcal{S}_1]\dot{\theta}_1 e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_n]\theta_n} M + \cdots + e^{[\mathcal{S}_1]\theta_1} \cdots [\mathcal{S}_n]\dot{\theta}_n e^{[\mathcal{S}_n]\theta_n} M\end{aligned}$$

- We have $T^{-1} = M^{-1} e^{-[\mathcal{S}_n]\theta_n} \cdots e^{-[\mathcal{S}_1]\theta_n}$
- Now recall that the velocity (twist) of the end-effector is $[\mathcal{V}_s] = \dot{T}T^{-1}$. We get

$$\begin{aligned}[\mathcal{V}_s] = [\mathcal{S}_1]\dot{\theta}_1 &+ e^{[\mathcal{S}_1]\theta_1}[\mathcal{S}_2]e^{-[\mathcal{S}_1]\theta_1}\dot{\theta}_2 \\ &+ e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2}[\mathcal{S}_3]e^{-[\mathcal{S}_2]\theta_2} e^{-[\mathcal{S}_1]\theta_1}\dot{\theta}_3 + \cdots\end{aligned}$$

Computing the Jacobian from the FKM in PoE form

- Using the adjoint map, we get

$$\mathcal{V}_s = \underbrace{\mathcal{S}_1}_{J_{s1}} \dot{\theta}_1 + \underbrace{\mathrm{Ad}_{e^{[\mathcal{S}_1]\theta_1}}(\mathcal{S}_2)}_{J_{s2}} \dot{\theta}_2 + \underbrace{\mathrm{Ad}_{e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2}}(\mathcal{S}_3)}_{J_{s3}} \dot{\theta}_3 + \cdots$$

- We see that $\mathcal{V}_s$ is the sum of n spatial twists. We set

$$J_s(\theta) = \begin{bmatrix} J_{s1} & J_{s2} & \cdots & J_{sn} \end{bmatrix}$$

  the Jacobian in space-frame coordinates, space Jacobian.
  We have

$$\mathcal{V}_s = J_s(\theta)\dot{\theta}$$

- For a n-joints mechanism, $J_s(\theta) \in \mathbb{R}^{6\times n}$.

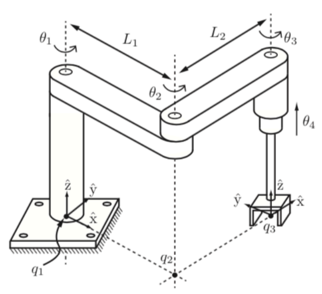
Computing the Jacobian from the FKM in PoE form

- The ith column of the space Jacobian is

$$J_{si} = \mathrm{Ad}_{e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_{i-1}]\theta_{i-1}}}(\mathcal{S}_i),$$

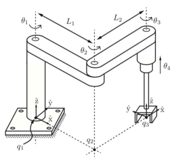
  for $i \geq 2$, and $J_{s1} = \mathcal{S}_1$
- Set $T_i = e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_i]}$. How to physically interpret this quantity?
- With M being "reference" configuration of the mechanism, $T_i M$ is the configuration when the first i joints are set to values $\theta_1, \ldots, \theta_i$ and the remaining are kept at zero $\longrightarrow$ T_i is the transformation matrix that takes the mechanism between the two states.
- J_{si} is the screw vector of joint i, in fixed frame coordinates, but expressed at arbitrary θ, (because of $\mathrm{Ad}_{T_{i-1}}$) instead of 0.
- Note that J_{si} only depends on $\theta_1 \cdots \theta_i$, we can ignore the other joint angles in its computation.

Computing the Jacobian: example



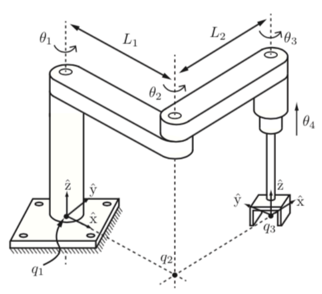
- To evaluate the space-Jacobian, we put in columns the screws of the joints in their order in the chain. We assume that the θ_i are arbitrary. We have here
  1. $\omega_{s1} = (0, 0, 1)$ and $v_{s1} = (0, 0, 0)$.
  2. $\omega_{s2} = (0, 0, 1)$. To compute v_{s2}, let q_2 be a vector joining origin of ref. frame to a point on axis of rotation of 2. For example, $q_2 = (L_1 \cos\theta_1, L_2 \sin\theta_1, 0)$. Then $v_{s2} = -\omega_2 \times q_2 = (L_1 s_1, -L_1 c_1, 0)$.

Computing the Jacobian: example



- For the other joints:
  1. $\omega_{s3} = (0, 0, 1)$. Choose q_3 joining origin to an arbitrary point on rotation axis, e.g. $q_3 = (L_1 c_1 + L_2 c_{12}, L_1 s_1 + L_2 s_{12}, 0)$, where $c_{12} = \cos(\theta_1 + \theta_2), s_{12} = \sin(\theta_1 + \theta_2)$, $c_1 = \cos\theta_1, s_2 = \sin\theta_2$. We get $v_{s3} = (L_1 s_1 + L_2 s_{12}, -L_1 c_1 - L_2 c_{12}, 0)$.
  2. The last joint is prismatic: $\omega_{s4} = (0, 0, 0)$. The axis of translation is always aligned with $\hat{z}$, regardless of the θ_i: $v_{s4} = (0, 0, 1)$.

Computing the Jacobian: example

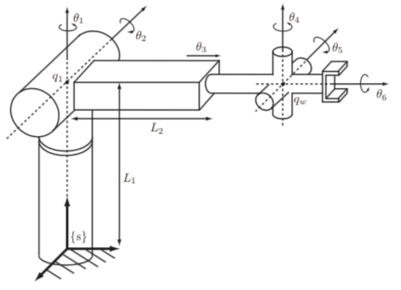


- The Jacobian is thus:

$$J_s(\theta) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & L_1 s_1 & L_1 s_1 + L_2 s_{12} & 0 \\ 0 & -L_1 c_1 & -L_1 c_1 - L_2 c_{12} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

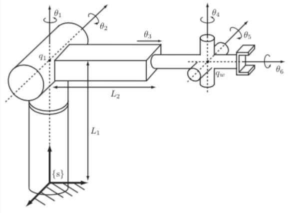
- If finding the screw vectors in arbitrary mechanism position is too difficult geometrically, use the formula derived earlier.

Computing the Jacobian: example



- First joint: $\omega_{s1} = (0,0,1)$, $q_1 = (0,0,L_1)$ and $v_{s1} = -\omega_1 \times q_1 = (0,0,0)$
- Second joint: axis in direction $\omega_{s2} = (-c_1, -s_1, 0)$. Set $q_2 = (0,0,L_1)$ and $v_{s2} = -\omega_{s2} \times q_2 = (L_1 s_2, -L_1 c_1, 0)$

Computing the Jacobian: example



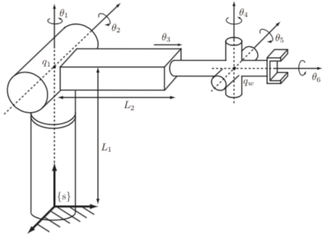
- Third joint: prismatic, so $\omega_{s3} = (0,0,0)$. The direction of motion at arbitrary positions of θ_1, θ_2 is

$$v_{s3} = Rot(\hat{z}, \theta_1) Rot(\hat{x}, -\theta_2) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -s_1 c_2 \\ c_1 c_2 \\ -s_2 \end{bmatrix}$$

- Fourth/fifth/sixth joints: also called wrist. Its center is at

$$q_w = \begin{bmatrix} 0 \\ 0 \\ L_1 \end{bmatrix} + Rot(\hat{z}, \theta_1) Rot(\hat{x}, -\theta_2) \begin{bmatrix} 0 \\ L_2 + \theta_3 \\ 0 \end{bmatrix} = \begin{bmatrix} -(L_2+\theta_3) s_1 c_2 \\ (L_2+\theta_3) c_1 c_2 \\ L_1 - (L_2+\theta_3) s_2 \end{bmatrix}$$

Computing the Jacobian: example

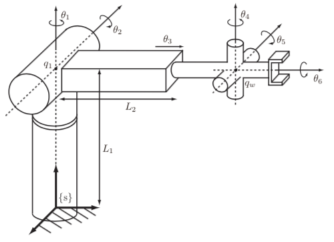


$$
\begin{aligned}
\omega_{s4} &= \mathrm{Rot}(\hat{z},\theta_1)\mathrm{Rot}(\hat{x},-\theta_2)\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}-s_1 s_2\\ c_1 s_2\\ c_2\end{bmatrix},\\
\omega_{s5} &= \mathrm{Rot}(\hat{z},\theta_1)\mathrm{Rot}(\hat{x},-\theta_2)\mathrm{Rot}(\hat{z},\theta_4)\begin{bmatrix}-1\\0\\0\end{bmatrix} = \begin{bmatrix}-c_1 c_4 + s_1 c_2 s_4\\ -s_1 c_4 - c_1 c_2 s_4\\ s_2 s_4\end{bmatrix}\\
\omega_{s6} &= \mathrm{Rot}(\hat{z},\theta_1)\mathrm{Rot}(\hat{x},-\theta_2)\mathrm{Rot}(\hat{z},\theta_4)\mathrm{Rot}(\hat{x},-\theta_5)\begin{bmatrix}0\\1\\0\end{bmatrix}\\
&= \begin{bmatrix}-c_5(s_1 c_2 c_4 + c_1 s_4) + s_1 s_2 s_5\\ c_5(c_1 c_2 c_4 - s_1 s_4) - c_1 s_2 s_5\\ -s_2 c_4 c_5 - c_2 s_5\end{bmatrix}.
\end{aligned}
$$

The direction of motion at arbitrary positions of θ_1, θ_2 is

$$
v_{s3} = Rot(\hat{z},\theta_1)Rot(\hat{x},-\theta_2)\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}-s_1 c_2\\ c_1 c_2\\ -s_2\end{bmatrix}
$$

Computing the Jacobian: example



- The Jacobian is

$$J_s(\theta) = \begin{bmatrix} \omega_{s1} & \omega_{s2} & 0 & \omega_{s4} & \omega_{s5} & \omega_{s6} \\ 0 & -\omega_{s2} \times q_2 & v_{s3} & -\omega_{s4} \times q_w & -\omega_{s5} \times q_w & -\omega_{s6} \times q_w \end{bmatrix}$$

Body Jacobian

- The (space) Jacobian relates the joint angles velocities to the end-effector twist in space-coordinates $[\mathcal{V}_s] = \dot{T}T^{-1}$, where $T = T_{sb}(\theta)$ is the position of the end-effector frame.
- Recall that the twist of the end-effector in body-frame is $[\mathcal{V}_b] = T^{-1}\dot{T}$, for T as above. We derived in the previous lectures that

$$T(\theta) = Me^{[\mathcal{B}_1]\theta_1} \cdots e^{[\mathcal{B}_n]\theta_n} = e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_n]\theta_n} M$$

- The body Jacobian is obtained by differentiating the first expression for $T(\theta)$.

Body Jacobian

- We have

$$\dot{T} = Me^{[\mathcal{B}_1]\theta_1} \cdots [\mathcal{B}_n]\dot{\theta}_n e^{[\mathcal{B}_n]\theta_n} + \cdots + M[\mathcal{B}_1]\dot{\theta}_1 e^{[\mathcal{B}_1]\theta_1} \cdots e^{[\mathcal{B}_n]\theta_n}$$

and

$$T^{-1} = e^{-[\mathcal{B}_n]\theta_n} \cdots e^{-[\mathcal{B}_1]\theta_1} M^{-1}$$

- Putting the two together, we get

$$[\mathcal{V}_b] = T^{-1}\dot{T} = [\mathcal{B}_n]\dot{\theta}_n + e^{-[\mathcal{B}_n]\theta_n}[\mathcal{B}_{n-1}]e^{-[\mathcal{B}_n]\theta_n}\dot{\theta}_{n-1} + \cdots \\ + e^{-[\mathcal{B}_n]\theta_n} \cdots e^{-[\mathcal{B}_2]\theta_2}[\mathcal{B}_1]e^{[\mathcal{B}_2]\theta_2} \cdots e^{[\mathcal{B}_n]\theta_n}\dot{\theta}_1$$

and in vector form

$$\mathcal{V}_b = \underbrace{\mathcal{B}_n}_{J_{bn}}\dot{\theta}_n + \underbrace{\mathrm{Ad}_{e^{-[\mathcal{B}_n]\theta_n}}(\mathcal{B}_{n-1})}_{J_{b,n-1}}\dot{\theta}_{n-1} + \cdots$$

Body Jacobian

- We conclude that

$$[\mathcal{V}_b] = \begin{bmatrix} J_{b1} & J_{b2} & \cdots & J_{bn} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \vdots \\ \dot{\theta}_n \end{bmatrix}$$

- The matrix J_b is the *body Jacobian*. The ith column of the body Jacobian is

$$J_{bi}(\theta) = \mathrm{Ad}_{e^{-[\mathcal{B}_n]\theta_n} \cdots e^{-[\mathcal{B}_{i+1}]\theta_{i+1}}}(\mathcal{B}_i)$$

for $i < n$, and $J_{bn} = \mathcal{B}_n$.

Relationship between space and body Jacobians

- We have $[\mathcal{V}_s] = \dot{T}_{sb} T_{sb}^{-1}$ and $[\mathcal{V}_b] = T_{sb}^{-1} \dot{T}_{sb}$. We have also shown that $\mathcal{V}_s = \mathrm{Ad}_{T_{sb}}(\mathcal{V}_b)$.
- We can also relate the twists to joint angle velocities through $\mathcal{V}_s = J_s(\theta)\dot{\theta}$ and $\mathcal{V}_b = J_b(\theta)\dot{\theta}$. Hence $\mathrm{Ad}_{T_{sb}}(\mathcal{V}_b) = J_s(\theta)\dot{\theta}$.
- Apply $\mathrm{Ad}_{T_{bs}}$ on both sides of the previous relation, and recall that $\mathrm{Ad}_A \mathrm{Ad}_B = \mathrm{Ad}_{AB}$

$$\mathrm{Ad}_{T_{bs}} \mathrm{Ad}_{T_{sb}}(\mathcal{V}_b) = \mathcal{V}_b = \mathrm{Ad}_{T_{bs}}(J_s(\theta)\dot{\theta}).$$

  Replace $\mathcal{V}_b$ by $J_b(\theta)\dot{\theta}$ to obtain $J_b(\theta)\dot{\theta} = \mathrm{Ad}_{T_{bs}}(J_s(\theta)\dot{\theta})$.
- Since the previous equation holds true for all $\dot{\theta}$, we have

$$J_b(\theta) = \mathrm{Ad}_{T_{bs}}(J_s(\theta)) \text{ and } J_s(\theta) = \mathrm{Ad}_{T_{sb}}(J_b(\theta))$$