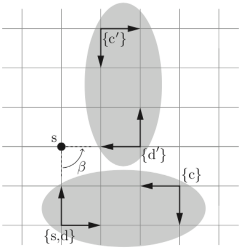
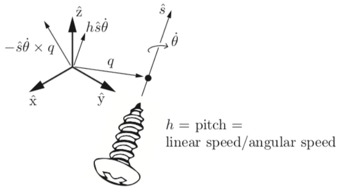
Introduction to Robotics
Lecture 7: Screws and wrenches

Screw motion



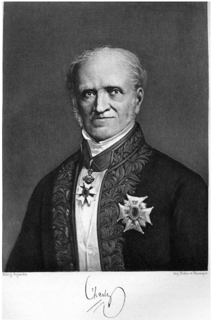
- 2D Screw motion: any rigid motion in the plane can be represented by a rotation around a well-chosen center.
- We can encode it with (β, s_x, s_y), where (s_x, s_y) is the position of the center of the rotation, and β the angle. Here, $\beta = \pi/4$ and $s = (0, 2)$

Screw motion



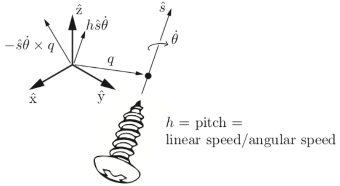
- **Chasles-Mozzi Theorem**: Any displacement in 3D can be represented by a rotation + translation about same axis. This is called a screw motion.
- Data for a screw: Axis of rotation: $q \in \mathbb{R}^3 + \hat{s} \in S^2$; pitch h : ratio linear/angular speed; $\dot{\theta}$: angular velocity.
- Rotation of θ rad results in translation of $h\theta$ along axis.
- Very useful to represent motion of revolute and prismatic joints.
- Write as $\mathcal{S} = \{q, \hat{s}, h\}$.

Ball and Chasles



- Rodrigues and Chasles took the entrance exam to Polytechnique/Normale at the same time, finishing first and second respectively. Rodrigues did not use it and elected to go to La Sorbonne.

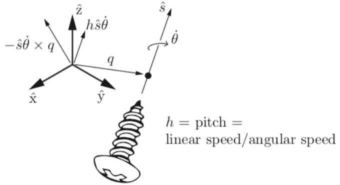
Screw motion → twist



- How to represent a screw motion as a twist? Take $\mathcal{S} = \{q, \hat{s}, h\}$.
  - Rotational velocity: $\omega = \hat{s}\dot{\theta}$.
  - Linear velocity : velocity at origin of frame= translation along the screw axis + linear motion at origin induced by rotation of screw axis:

$$v = h\hat{s}\dot{\theta} - \hat{s}\dot{\theta} \times q.$$

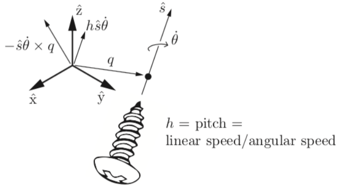
Twist $\rightarrow$ Screw motion



- How to represent a twist as a screw motion? Take $\mathcal{V} = [\omega, v]^\top$
  - Rotation+pitch: $\hat{s} = \omega/\|\omega\| = \bar{\omega}$, $\dot{\theta} = \|\omega\|$ and $h = \bar{\omega}^\top v/\dot{\theta} = \omega^\top v/\|\omega\|^2$.
  - Offset q: Assuming $\omega \neq 0$ (not pure translation, so that there is a meaningful rotation axis), need to find q so that $v = h\hat{s}\dot{\theta} - \hat{s}\dot{\theta} \times q$. Cross product on left by $\hat{s}$ yields

$$q = \frac{\hat{s} \times v}{\dot{\theta}}.$$

Screw motion



- For a given reference frame, we denote by $\mathcal{S}$ the *screw axis* of the screw motion $\{q, \hat{s}, \dot{\theta}\}$

$$\mathcal{S} = \begin{bmatrix} \omega \\ v \end{bmatrix}$$

with either 1: $\|\omega\| = 1$ or 2: $\omega = 0$ and $\|v\| = 1$.
Case 1: then $v = -\omega \times q + h\omega$
Case 2: corresponds to a translation along v: (h is infinite) $\rightarrow \omega = 0$.

Exponential coordinates for Rigid-Body motions

- Recall: for a rotation $R \in SO(3)$, its *exponential coordinates* $(\bar{\omega}, \theta)$ are so that $R = e^{[\bar{\omega}]\theta}$. We want to do the same for rigid motions (i.e., rotation + translation)
- The matrix $[\bar{\omega}]$ was skew-symmetric: $[\bar{\omega}] \in \mathfrak{so}(3)$.
- We defined

$$\exp : \mathfrak{so}(3) \to SO(3) : A \to I + A + A^2/2! + \cdots$$
$$\log : SO(3) \to \mathfrak{so}(3) : R \to \frac{1}{2 \sin \theta}(R - R^\top)$$

Exponential coordinates for Rigid-Body motions

- By analogy, we define the exponential coordinates for a homogeneous transformation T as $\mathcal{S}\theta \in \mathbb{R}^6$, where $\mathcal{S}$ is the screw axis and θ the distance around the screw axis to take a frame from I to T.
- Recall that if $\mathcal{S} = (\omega, v)$ is with $\|\omega\| = 1$, then θ is an angle of rotation about the screw axis. If $\omega = 0$ and $\|v\| = 1$, then θ is the distance travelled along the screw axis.
- Denote by $SE(3)$ the space of homogeneous transformations T, and by $\mathfrak{se}(3)$ the space of their exponential coordinates. We want to define

$$\exp : \mathfrak{se}(3) \to SE(3) : A \to I + A + A^2/2! + \cdots$$
$$\log : SE(3) \to \mathfrak{se}(3) : T \to [\mathcal{S}]\theta \in \mathfrak{se}(3)$$

Exponential coordinates for Rigid-Body motions

For $\mathcal{S} = [\omega, v]^\top$, we introduce the 4×4 matrix:

$$[\mathcal{S}] := \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix}$$

- As before, we can find a simple expression for the exponential. We have (recall: $\omega = \overline{\omega}\theta$)

$$\begin{aligned} e^{[\mathcal{S}]\theta} &= I + [\mathcal{S}]\theta + [\mathcal{S}]^2\theta^2/2! + [\mathcal{S}]^3\theta^3/3! + \cdots \\ &= \begin{bmatrix} e^{[\overline{\omega}]\theta} & G(\theta)v \\ 0 & 1 \end{bmatrix} \end{aligned}$$

  where $G(\theta) = I\theta + [\overline{\omega}]\theta^2/2! + [\overline{\omega}]^2\theta^3/3! + \cdots$.
- Recall that $[\overline{\omega}]^3 = -[\overline{\omega}]$. We obtain

$$\begin{aligned} G(\theta) &= I\theta + (\theta^2/2! - \theta^4/4! + \cdots)[\overline{\omega}] + (\theta^3/3! - \theta^5/5! + + \cdots)[\overline{\omega}]^2 \\ &= I\theta + (1 - \cos\theta)[\overline{\omega}] + (\theta - \sin\theta)[\overline{\omega}]^2 \end{aligned}$$

Logarithm of Rigid-Body motions

- Given $T = (R, p) \in SE(3)$, we need to find $\mathcal{S} = (\overline{\omega}, v)$ and θ so that

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix},$$

  which implies that

$$[\mathcal{S}]\theta = \begin{bmatrix} [\overline{\omega}]\theta & v\theta \\ 0 & 0 \end{bmatrix}$$

  is a logarithm of T.
- Case 1: If $R = I$ (pure translation), then set $\overline{\omega} = 0$, $v = p/\|p\|$ and $\theta = \|p\|$.
- Case 2: **1.** Obtain $\overline{\omega}, \theta$ using the matrix logarithm of R. **2.** Set

$$v = G^{-1}(\theta)p$$

  where

$$G^{-1}(\theta) = \frac{1}{\theta}I - \frac{1}{2}[\overline{\omega}] + \left(\frac{1}{\theta} - \frac{1}{2}\cot\frac{\theta}{2}\right)[\overline{\omega}]^2.$$

Wrenches

- Let $\{a\}$ be a ref. frame and r_a a point in a rigid body. Let the vector $f_a \in \mathbb{R}^3$ represent a force acting on the body. Assume it is a point force acting at r_a.
- The force creates a *torque or moment* $m_a \in \mathbb{R}^3$ given by

$$m_a := r_a \times f_a.$$

- We merge the force and the moment in a single 6-dimensional vector, called *spatial force* or *wrench*, expressed in $\{a\}$ as

$$\mathcal{F}_a := \begin{bmatrix} m_a \\ f_a \end{bmatrix}.$$

- A wrench can contain only a torque and no force: it is called a *pure moment*.
- If there are several forces/moments acting on the body, we can add their corresponding wrenches.

Wrenches

- Given 2 frames, *a* and *b*, and want to express the wrench in both frames: $\mathcal{F}_a$, $\mathcal{F}_b$.
- Denote by $\mathcal{V}_a$, $\mathcal{V}_b$ the motions induced by the wrench in frames *a* and *b*: we know how to relate them: $\mathcal{V}_b = \mathrm{Ad}_{T_{ba}} \mathcal{V}_a$
- Recall: dot product of velocity and force is a power!
- Power is independent from choice of frame!

$$
\begin{aligned}
\mathcal{V}_b^\top \mathcal{F}_b &= \mathcal{V}_a^\top \mathcal{F}_a \\
&= (\mathrm{Ad}_{T_{ab}} \mathcal{V}_b)^\top \mathcal{F}_a \\
&= \mathcal{V}_b^\top (\mathrm{Ad}_{T_{ab}})^\top \mathcal{F}_a
\end{aligned}
$$

- Since the previous equation holds for *all* $\mathcal{V}_b$, we have

$$
\mathcal{F}_b = (\mathrm{Ad}_{T_{ab}})^\top \mathcal{F}_a
$$