Introduction to Robotics
Lecture 5: Homogeneous transformations and angular velocities

Homogeneous transformation matrices

Definition

The **special euclidean group** $SE(3)$ is the set of 4×4 matrices of the form

$$T = T(R, p) = \begin{pmatrix} R & p \\ \mathbf{0} & 1 \end{pmatrix},$$

where $R \in SO(3)$, $p \in \mathbb{R}^3$ and $\mathbf{0} = (0,\ 0,\ 0)$.

- The inverse of T is

$$T^{-1} = \begin{pmatrix} R^\top & -R^\top p \\ \mathbf{0} & 1 \end{pmatrix}.$$

  In particular, $T^{-1} \in SE(3)$.
- If $T_1, T_2 \in SE(3)$, then $T_1 T_2 \in SE(3)$

Properties of homogeneous transformation matrices

- If $T_1, T_2, T_3 \in SE(3)$, then $(T_1 T_2) T_3 = T_1 (T_2 T_3)$: associative. We can choose which pairwise product to do first, but we cannot change the order! ($T_1 T_2 \neq T_2 T_1$ in general).
- We have $T(R, p) \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} Rx + p \\ 1 \end{pmatrix}$
- We call $\begin{pmatrix} x \\ 1 \end{pmatrix} \in \mathbb{R}^4$ the *homogeneous* representation of $x \in \mathbb{R}^3$.
- T satisfies the following relations:

$$\|Tx - Ty\| = \|x - y\|$$
$$(Tx - Tz)^\top (Ty - Tz) = (x - z)^\top (y - z)$$

$\Longrightarrow$ whence the name *rigid motion*: preserves lengths and angles, and thus the shape of solid objects.

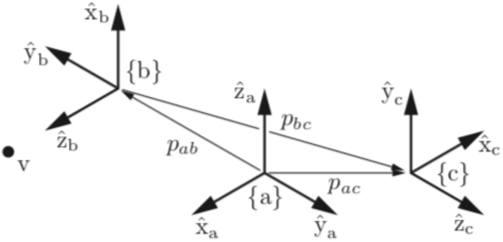
Use of homogeneous transformation matrices

Transformation matrices $T \in SE(3)$ can be used for

1. represent the configuration (position+orientation) or a rigid body in 3D
2. change the reference frame with respect to which a configuration is described
3. move a frame to describe motions.

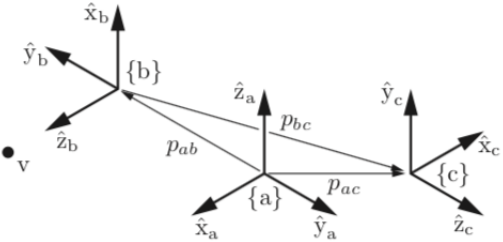
This is similar to what we have described in the planar case. We describe these three uses in the next slides.

Representing a configuration with $SE(3)$



- Given a multi-link mechanism, with reference frames a and body frames b and c.
- We can describe the orientation/position (R_{sb}, p_{sb}) of b with respect to s, and the orientation/position (R_{bc}, p_{bc}) of c with respect to b.
- Create $T_1 = T(R_{sb}, p_{sb})$ and $T_2 = T(R_{bc}, p_{bc})$.

Changing reference frame for a vector/frame



- To change reference frames, we have the rules

$$T_{ab}T_{bc} = T_{ac}$$
$$T_{ab}v_b = v_a$$

Displacing frames/vectors

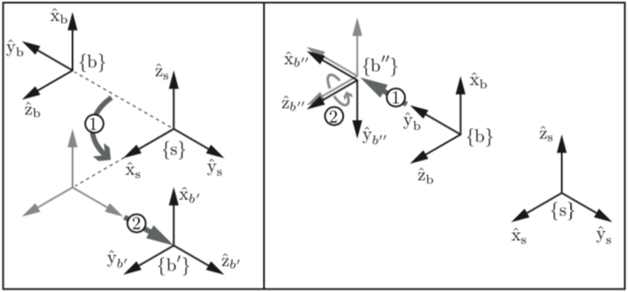
- Given a frame T_{sb}, we want to rotate it by a rotation $R = (\bar{\omega}, \theta)$ and translate it by a vector p. This is useful for, e.g., describing motion of links.
- We need to pay attention to *what frame* the rotation/translation are described with respect to, and whether we apply the rotation or the translation first.
- Define

$$Rot(\bar{\omega}, \theta) = \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} \text{ and } Trans(p) = \begin{pmatrix} I & p \\ 0 & 1 \end{pmatrix},$$

  and as usual

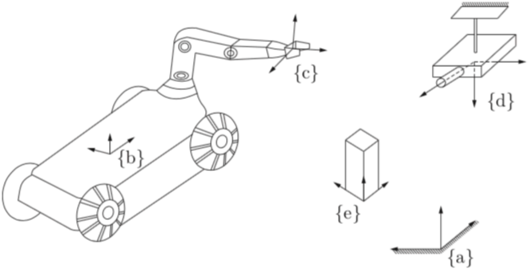
$$T(R, p) = \begin{pmatrix} R & p \\ 0 & 1 \end{pmatrix}.$$

Displacing frames/vectors



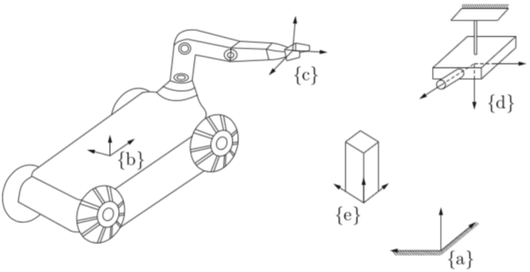
- Multiply T_{sb} on left by Rot/Trans/T: interpreted in the s frame.
- Multiply T_{sb} on right by Rot/Trans/T: interpreted in the b frame.
- Action "closest" to T_{sb} is executed first (rotation or translation). For left-multiplication, rotate first then translate since $T(R, p) = Trans(p)Rot(\bar{\omega}, \theta)$.
- Example: $\bar{\omega} = (0, 0, 1)$, $p = (0, 2, 0)$ and $\theta = 90$.

Displacing frames/vectors: example



- Robot arm mounted on wheeled platform. Camera fixed to ceiling. b is body frame, c end-effector frame, e object to pick up frame. Fixed ref. frame is a. We assume T_{ad} known (camera position in ref. frame)
- From camera measurements, you can evaluate T_{db}. T_{bc} can be calculated using joint-angle measurements.
- To pick up object, we need position of object w/ respect end-effector: T_{ce}.

Displacing frames/vectors: example



- Goal: get T_{ce}. We have the relation

$$T_{ab}T_{bc}T_{ce} = T_{ae} = T_{ad}T_{de}.$$

  We need T_{ab}, the position of the wheelbase with respect to the body frame. We obtain it as $T_{ab} = T_{ad}T_{db}$.
- We thus obtain

$$T_{ce} = (T_{ad}T_{db}T_{bc})^{-1}T_{ad}T_{de}.$$

Linear and angular velocities of a moving frame

- Let $\{s\}$ and $\{b\}$ be the reference and body frames respectively. We want to express angular and linear velocities in these frames.
- Let $T_{sb}(t) = T(t) = \begin{bmatrix} R(t) & p(t) \\ 0 & 1 \end{bmatrix}$. We see that

$$\dot{T} = \begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix}.$$

- Premultiplying by T^{-1} we obtain

$$\begin{aligned} T^{-1}\dot{T} &= \begin{bmatrix} R^\top & -R^\top p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} R^\top \dot{R} & R^\top \dot{p} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} [\omega_b] & v_b \\ 0 & 0 \end{bmatrix} \end{aligned}$$

  Note that $\dot{p}$ is the linear velocity of the origin of b in s, and v_b is this velocity expressed in b.

Body twists

- The matrix $T^{-1}\dot{T}$ has 6 free parameters (3 for ω_b and 3 for v_b.
- We gather them in a vector

$$\mathcal{V}_b = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix}.$$

  We call $\mathcal{V}_b$ the **spatial velocity in the body frame**, or **body twist**.
- We also use the *bracket notation*

$$[\mathcal{V}_b] := \begin{bmatrix} [\omega_b] & v_b \\ 0 & 0 \end{bmatrix},$$

  where we recall that $[\omega_b]$ is a skew-symmetric matrix (but $[\mathcal{V}_b]$ is not!).

Linear and angular velocities in ref. frame

- We saw that, in the case of angular velocities, post-multiplying by R^{-1} yields the angular velocity in the reference frame. Let us see what happens here:

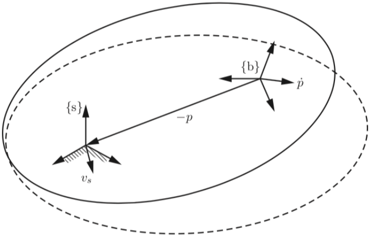
$$\dot{T}T^{-1} = \begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^\top & -R^\top p \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \dot{R}R^\top & \dot{p} - \dot{R}R^\top p \\ 0 & 0 \end{bmatrix} =: \begin{bmatrix} [\omega_s] & v_s \\ 0 & 0 \end{bmatrix}$$

- We have seen earlier that $\dot{R}R^\top = [\omega_s]$ is the angular velocity in the frame $\{s\}$.
- The linear velocity in frame s is however simply $\dot{p}$ (by definition!).
- Let us try to interpret v_s. We have

$$v_s = \dot{p} - [\omega_s]p = \dot{p} - \omega_s \times p.$$

Spatial twists



- We have $v_s = \dot{p} - [\omega_s]p = \dot{p} + \omega_s \times (-p)$.
- v_s is *not* the linear velocity of the body frame origin in the fixed frame (that is $\dot{p}$.)
- Assume that a very large rigid body is attached to the frame b, large enough to contain the origin of s. *What is the velocity of the point of this body as the origin of s?* There are 2 components: $\dot{p}$, the motion of the body, and $\omega_s \times (-p)$, due to the rotation of the body $\longrightarrow$ this is v_s.

Twists

- We can relate $[\mathcal{V}_s]$ and $[\mathcal{V}_b]$ through

$$[\mathcal{V}_b] = T^{-1}[\mathcal{V}_s]T \text{ and } [\mathcal{V}_s] = T[\mathcal{V}_b]T^{-1}.$$

  These formulas are obtained by replacing $\dot{T}$ by its expression in terms of $[\mathcal{V}_s]$ or $[\mathcal{V}_b]$.
- We now look to express the above relations directly in terms of $\mathcal{V}_s$ and $\mathcal{V}_b$.
- Writing out the products above, we get

$$\mathcal{V}_s = \begin{bmatrix} R[\omega_b]R^\top & -R[\omega_b]R^\top p + Rv_b \\ 0 & 0 \end{bmatrix}$$

- Using $R[\omega]R^\top = [R\omega]$ and the relation $[\omega]p = \omega \times p = -p \times \omega = -[p]\omega$, we can simplify the above to obtain

$$\begin{bmatrix} \omega_s \\ v_s \end{bmatrix} = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \begin{bmatrix} \omega_b \\ v_b \end{bmatrix}$$

Adjoint representation

- For future use, we record the definition:

$$\mathrm{Ad}_T = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \in \mathbb{R}^{6\times 6}$$

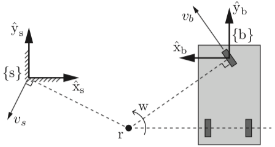
  is called the *adjoint representation* of $T = (R, p) \in SE(3)$.
- We call $\mathrm{Ad}_T(\mathcal{V}) := \mathrm{Ad}_T\mathcal{V}$ the adjoint map of T. By construction, we have

$$\mathcal{V}' = \mathrm{Ad}_T\mathcal{V} \Leftrightarrow [\mathcal{V}'] = T[\mathcal{V}]T^{-1}.$$

- The adjoint map has the easily verified properties:

$$\mathrm{Ad}_{T_1}\mathrm{Ad}_{T_2} = \mathrm{Ad}_{T_1 T_2} \text{ and } \mathrm{Ad}_{T^{-1}} = (\mathrm{Ad}_T)^{-1}.$$

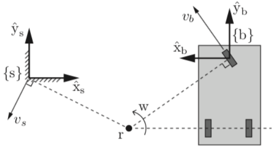
Example



- Motion is pure angular velocity of 2 rad/s around r. We have $r_s = (2, -1, 0)^\top$, $r_b = (2, -1.4, 0)^\top$, $\omega_s = (0, 0, 2)^\top$ and $\omega_b = (0, 0, -2)^\top$. The position of the car frame is

$$T_{sb} = \begin{bmatrix} R_{sb} & p_{sb} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0.4 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Example



- Recall that v_x is the velocity of the point at the origin of frame x. If the origin is not in the body, assume the body to be infinitely large.
- $v_b = \omega_b \times (-r_b) = (2.8, 4, 0)$
- $v_s = \omega_s \times (-r_s) = (-2, -4, 0)$
- What is v at a frame with axes parallel to x_s, y_s, z_s and centered at r?
- You can verify that $\mathcal{V}_s = [\omega_s, v_s] = [\mathrm{Ad}_{T_{sb}}]\mathcal{V}_b$.