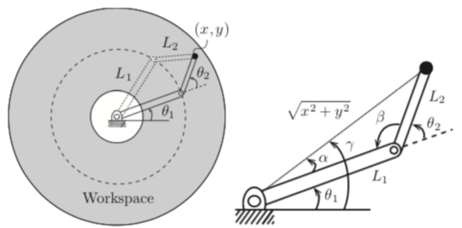
Introduction to Robotics
Lecture 11: Inverse Kinematics

Inverse kinematics



(a) A workspace, and lefty and righty configurations. (b) Geometric solution.

- Forward kinematics: compute the end-effector position (as an element of $SE(3)$) from joint angles θ_i: compute the function

$$T : \text{joint space} \to SE(3) : \theta \mapsto T(\theta)$$

- Inverse kinematics: compute the (possible) joint angles from the position of the end-effector: compute the function

$$T^{-1} : SE(3) \to \text{joint space} : X \mapsto \theta.$$

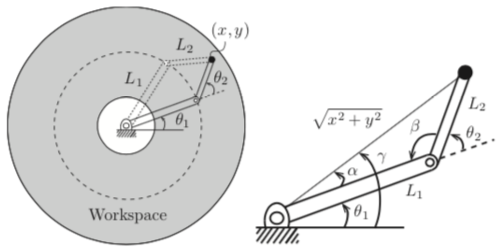
- The inverse kinematics function is often *multi-valued*.

The two argument arctan function: atan2

- Returns the angle between x-axis and vector (x, y) in the plane.
- Unline atan, which is valued in $(-\pi/2, \pi/2]$, atan2 is valued in $(-\pi, \pi]$.
- It is a default trig. function in most programming languages.

$$\text{atan2}(y, x) = \begin{cases} \text{atan}(y/x) \text{ if} & x > 0 \\ \text{atan}(y/x) + \pi \text{ if} & x < 0 \text{ and } y \geq 0 \\ \text{atan}(y/x) - \pi \text{ if} & x < 0 \text{ and } y < 0 \\ \pi/2 \text{ if} & x = 0 \text{ and } y > 0 \\ -\pi/2 \text{ if} & x = 0 \text{ and } y < 0 \\ \text{undefined if} & x = 0 \text{ and } y = 0 \end{cases}$$

Analytic inverse kinematics

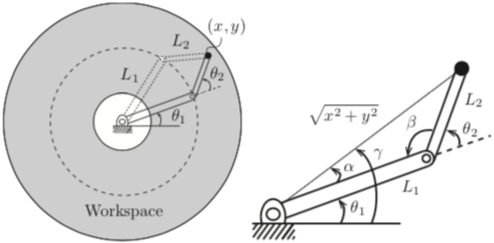


(a) A workspace, and lefty and righty configurations.

(b) Geometric solution.

- Recall: Law of cosines
$$c^2 = a^2 + b^2 - 2ab\cos(\gamma),$$
where a, b, c are the lengths of the edges of the triangle, and α, β, γ the angles opposite a, b and c respectively.
- We have $L_1^2 + L_2^2 - 2L_1L_2\cos\beta = x^2 + y^2$. It follows
$$\beta = \cos^{-1}\left(\frac{L_1^2 + L_2^2 - x^2 - y^2}{2L_1L_2}\right)$$

Analytic inverse kinematics



(a) A workspace, and lefty and righty configurations.

(b) Geometric solution.

- Similarly, $\alpha = \cos^{-1}\left(\frac{L_1^2 - L_2^2 + x^2 + y^2}{2L_1\sqrt{x^2+y^2}}\right)$
- Using atan2 function, we get $\gamma = \text{atan2}(y, x)$.
- The two possible solutions are

$$\theta_1 = \gamma - \alpha, \theta_2 = \pi - \beta \text{ and } \theta_1 = \gamma + \alpha, \theta_2 = \beta - \pi$$

- If $x^2 + y^2 \notin [L_1 - L_2, L_1 + L_2]$, then no solutions exist.

Analytic inverse kinematics

- In the 2R robot example, there were 2 DOFs for the end-effector, and 2 joint angles. This implied that there was a *finite* number of solutions.
- If there are more joint angles than DOFs of the end-effector, there may be an *infinite number of solutions*.
- We will mostly look at cases where $\#$DOFs of end-effector $= \#$ joint angles.
- We thus assume in general

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_6]\theta_6} M$$

and we are given an end-effector pose $X \in SE(3)$. We need to find $T^{-1}(X)$.

Analytic inverse kinematics: Euler angles

- Euler angles are useful in evaluating inverse kinematic maps analytically.
- The ZYX Euler angles can be used to represent an arbitrary rotation in $\mathbb{R}^3$ as follows:

$$R(\alpha, \beta, \gamma) = Rot(\hat{z}, \alpha) Rot(\hat{y}, \beta) Rot(\hat{x}, \gamma)$$

with $\alpha, \gamma \in (-\pi, \pi]$ and $\beta \in [-\pi/2, \pi/2)$ and

$$\mathrm{Rot}(\hat{z}, \alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathrm{Rot}(\hat{y}, \beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix},$$

$$\mathrm{Rot}(\hat{x}, \gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}.$$

Analytic inverse kinematics: Euler angles

- We now look at the inverse problem: given $R \in SO(3)$, can we always find α, β, γ so that $R(\alpha, \beta, \gamma) = R$? The answer is yes, and we now show how:
- Explicitly, we have

$$R(\alpha, \beta, \gamma) = \begin{bmatrix} c_\alpha c_\beta & c_\alpha s_\beta s_\gamma - s_\alpha c_\gamma & c_\alpha s_\beta c_\gamma + s_\alpha s_\gamma \\ s_\alpha c_\beta & s_\alpha s_\beta s_\gamma + c_\alpha c_\gamma & s_\alpha s_\beta c_\gamma - c_\alpha s_\gamma \\ -s_\beta & c_\beta s_\gamma & c_\beta c_\gamma \end{bmatrix}$$

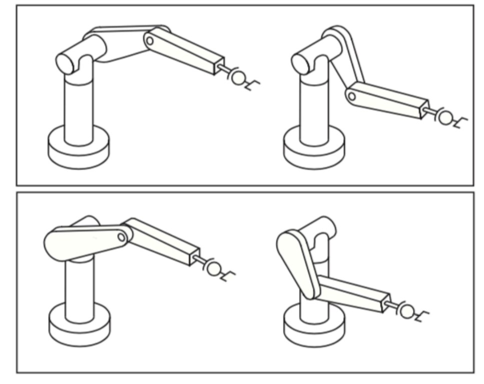
- Denote by r_{ij} the $ij$the entry of R. We can first look at r_{31} and articulate our answer around its value

Analytic inverse kinematics: Euler angles

$$R(\alpha,\beta,\gamma) = \begin{bmatrix} c_\alpha c_\beta & c_\alpha s_\beta s_\gamma - s_\alpha c_\gamma & c_\alpha s_\beta c_\gamma + s_\alpha s_\gamma \\ s_\alpha c_\beta & s_\alpha s_\beta s_\gamma + c_\alpha c_\gamma & s_\alpha s_\beta c_\gamma - c_\alpha s_\gamma \\ -s_\beta & c_\beta s_\gamma & c_\beta c_\gamma \end{bmatrix}$$

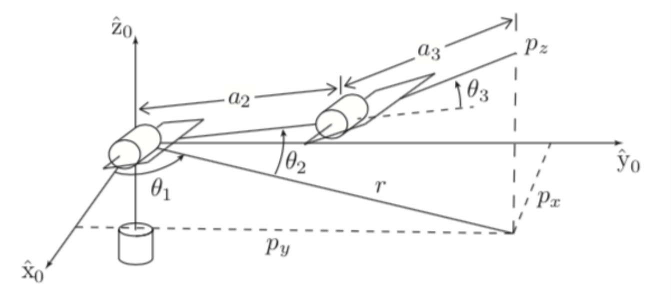
1. If $r_{31} \neq \pm 1$ set $\beta = \text{atan2}(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2})$, $\alpha = \text{atan2}(r_{21}, r_{11})$ and $\gamma = \text{atan2}(r_{32}, r_{33})$.
2. If $r_{31} = -1$, then $\beta = \pi/2$. There exists an infinite number of solutions for α and γ. One such solution is $\alpha = 0, \gamma = \text{atan2}(r_{12}, r_{22})$
3. If $r_{31} = 1$, then $\beta = -\pi/2$. There exists an infinite number of solutions for α and γ. One such solution is $\alpha = 0, \gamma = -\text{atan2}(r_{12}, r_{22})$

Analytic inverse kinematics: 6R Puma arm



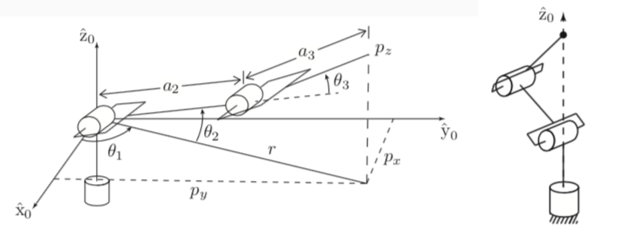
- PUMA stands for Programmable Universal Machine for Assembly
- Industrial robot arm, developed for car manufacturing in late 1970's. Still widely used today.

Analytic inverse kinematics: 6R Puma arm

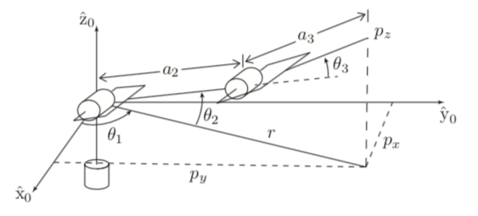


Zero position:

1. Two shoulder joint intersect orthogonally at a common point. Joint axis 1 is aligned with $\hat{z}_0$, joint axis 2 with $\hat{y}_0$.
2. Joint axis 3 (elbow) in $\hat{x}_0, \hat{y}_0$ plane and parallel with joint axis 2
3. Joint 4-5-6 form a wrist. They intersect orthogonally at a common point and are aligned to the $\hat{z}_0, \hat{y}_0$ and $\hat{x}_0$ directions respectively.



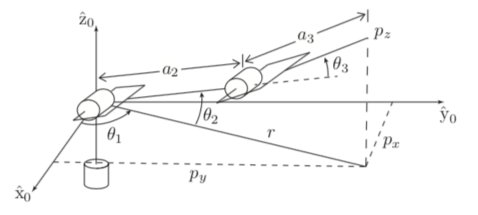
- The inverse kinematics problem can be split into inverse orientation and position problems (Not true for all mechanisms!)
- Let $p = (p_x, p_y, p_z)$ be position of the wrist center.
- Assume $(p_x, p_y) \neq (0, 0)$ We have that $\theta_1 = \text{atan2}(p_y, p_x)$.
- When $(p_x, p_y) = (0, 0)$, we are in a singular configuration, there are infinitely many solutions for θ_1.



- Finding θ_2 and θ_3 reduces to IK for planar 2R robot.
- Applying what we had derived before to this case, we get

$$\cos\theta_3 = (r^2 + p_z^2 - a_2^2 - a_3^2)/(2a_2a_3) = D$$

We then have $\theta_3 = \text{atan2}(\sqrt{1-D^2}, D)$



- We obtain for θ_2

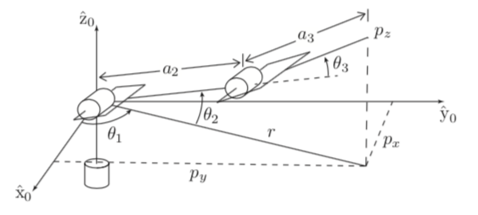
$$\theta_2 = \text{atan2}(p_z, r) - \text{atan2}(a_3 s_3, a_2 + a_3 c_3)$$

- Recall that $X = e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_6]\theta_6} M$, and we know M and X and have just figured out what $\theta_1, \theta_2, \theta_3$ are. We thus need to solve

$$e^{[\mathcal{S}_4]\theta_4} e^{[\mathcal{S}_5]\theta_5} e^{[\mathcal{S}_6]\theta_6} = e^{[-\mathcal{S}_3]\theta_3} e^{-[\mathcal{S}_2]\theta_2} e^{-[\mathcal{S}_1]\theta_1} X M^{-1}$$

where $\omega_4 = (0, 0, 1), \omega_5 = (0, 1, 0)$ and $\omega_6 = (1, 0, 0)$.

Analytic inverse kinematics: 6R Puma arm



- Denote by R the rotation component of $e^{[-\mathcal{S}_3]\theta_3}e^{-[\mathcal{S}_2]\theta_2}e^{-[\mathcal{S}_1]\theta_1}XM^{-1}$. We thus need to find $\theta_4, \theta_5, \theta_6$ so that

$$Rot(\hat{z}, \theta_4)Rot(\hat{y}, \theta_5)Rot(\hat{x}, \theta_6) = R.$$

  This is exactly the ZYX Euler angles problem we have solved.