Introduction to Robotics
Lecture 10: Statics of open chains, Singularities and Manipulability

Statics of open chains

- Considering the robot to be at static equilibrium, we can equate power at the joint with power at the end-effector:

$$\tau^\top \dot{\theta} = \mathcal{F}_b \mathcal{V}_b$$

- Since $\mathcal{V}_b = J_b(\theta)\dot{\theta}$, we get $\tau = J_b^\top(\theta)\mathcal{F}_b$ and $\tau = J_s^\top(\theta)\mathcal{F}_s$. We simply write

$$\tau = J^\top(\theta)\mathcal{F}$$

- We can use the above equation to derive the torques to apply at the joint to counteract the effect of an external wrench $-\mathcal{F}$ applied at the end-effector (e.g. by a load attached to it)
- Recall $J(\theta) \in \mathbb{R}^{6\times n}$. Depending on whether $n > 6$ or $n < 6$, the system is over- or under-determined.

Statics of open chains

- If $n > 6$, the chain is called redundant. Immobilizing the end-effector does not fix the value of the torque joints.
- We need to consider dynamics to continue the analysis. We do not do it here.
- If $n < 6$ and $J^\top \in \mathbb{R}^{n \times 6}$, then we do not have enough joints to generate 6 DoFs of the end-effector twist. We cannot *actively* generate forces in the $6 - n$ directions in the nullspace of $J^\top(\theta)$

Singularity analysis

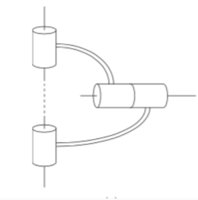
- Values of the joints θ_i for which the end-effector cannot move in one or more directions instantaneously is called a *kinematic singularity*
- Mathematically, this corresponds to θ for which $J(\theta)$ is not full (column) rank.
- Since J has 6 rows, it means that $J(\theta)$ has fewer that 6 linearly independent columns at a singular configuration.
- Said otherwise: there are 6 degrees of freedom for the twist of the end-effector, and each column of the Jacobian maps into some possible motions for the end-effector. To be non-singular (i.e. regular), we need at least 6 independent possible motions.

$$\mathcal{V}_b = J_{b1}\dot{\theta}_1 + \cdots + J_{bn}\dot{\theta}_n.$$

Singularity analysis

- Kinematic singularities are independent of the frame chosen to express the twist. Indeed, recall that $\mathcal{V}_s = \mathrm{Ad}_{T_{sb}} \mathcal{V}_b$ and thus
$$J_s = \mathrm{Ad}_{T_{sb}} J_b.$$
- Since $\mathrm{Ad}_{T_{sb}}$ is invertible, we conclude that J_s and J_b have the same rank.

Examples of singular configurations

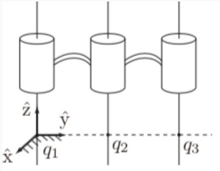


- We assume that there are $n = 6$ joints with 1DOF each. The Jacobian has thus 6 columns.
- Assume 2 R joints, say 1 and 2, have collinear revolution axes: $\omega_{s1} = \pm\omega_{s2}$.
- Then the corresponding column of the Jacobian are

$$J_{s_1} = \begin{bmatrix} \omega_{s1} \\ -\omega_{s1} \times q_1 \end{bmatrix} \text{ and } J_{s_2} = \begin{bmatrix} \omega_{s2} \\ -\omega_{s2} \times q_2 \end{bmatrix}.$$

- Since $q_1 - q_2$ is aligned with ω_{si}, we have $\omega_{si} \times (q_1 - q_2) = 0$ and thus $\omega_{si} \times q_1 = \omega_{si} \times q_2$. Hence $J_{b1} = J_{b2}$. The Jacobian is thus singular, for *any* θ.

Examples of singular configurations



- Three coplanar and parallel revolute joint axes. The first three columns of the Jacobian are

$$J_s(\theta) = \begin{bmatrix} \omega_{s1} & \omega_{s1} & \omega_{s1} & \cdots \\ 0 & -\omega_{s1} \times q_2 & -\omega_{s1} \times q_3 & \cdots \end{bmatrix}$$

- Since q_2 and q_3 are aligned, so are $-\omega_{s1} \times q_2$ and $-\omega_{s1} \times q_3$. The Jacobian cannot be regular.