Introduction to Robotics
Lecture 4: 3D rotations and exponential coordinates

Exponential coordinates

- We now introduce a 3-parameters representation for 3D rotations [instead of the 3×3 orthogonal matrix].
- Idea: represent the rotation through a rotation axis $\hat{\omega}$ (normalized, i.e. $\|\hat{\omega}\| = 1$) and a rotation angle around this axis, say θ. The vector $\omega = \hat{\omega}\theta$ contains the three-parameters exponential coordinate representation of the rotation.
- If a frame coincident with s is rotated for 1 second around $\hat{\omega}$ at angular velocity θ, then the resulting frame is R.
- Equivalently, if a frame coincident with s is rotated for θ seconds around $\hat{\omega}$ at angular velocity 1, then the resulting frame is R.

Review from Linear ODEs

- Consider the scalar linear ODE
$$\dot{x} = ax(t)$$
with initial state $x(0) = x_0$. Its solution at time t is
$$x(t) = e^{at}x_0.$$
- The exponential function has the expansion
$$e^{at} = 1 + at + \frac{1}{2}a^2t^2 + \frac{1}{3!}a^3t^3 + \cdots .$$
- Consider the *vector* linear ODE
$$\dot{x} = Ax,$$
with $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$ and $x(0) = x_0$.
- We can write its solution as
$$x(t) = e^{At}x_0$$
where
$$e^{At} = 1 + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots$$

Some properties of matrix exponential

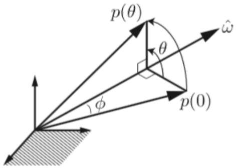
The matrix exponential

$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots$$

has the following properties

- $\frac{d}{dt}(e^{At}) = Ae^{At} = e^{At}A$.
- If $A = PDP^{-1}$ with D a *diagonal matrix*, then $e^{At} = Pe^{Dt}P^{-1}$
- If A and B *commute*, i.e. $AB = BA$, then $e^Ae^B = e^Be^A = e^{A+B}$
- The matrix exponential of A is *always* invertible, and $(e^{At})^{-1} = e^{-At}$.

Some properties of matrix exponential



- Assume the vector $p(0)$ is rotated by θ around $\hat{\omega}$ to $p(\theta)$.
- We can assume that $p(t)$ rotates at a constant rate of 1 rad/s for a time θ. We thus have

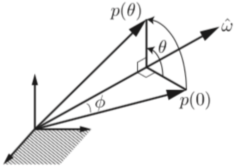
$$\dot{p} = \hat{\omega} \times p$$

  for θ seconds.
- We can write this equation as

$$\dot{p} = [\hat{\omega}]p,$$

  whose solution is $p(t) = e^{[\hat{\omega}]t}p(0)$.
- We conclude that $p(\theta) = e^{[\hat{\omega}]\theta}p(0)$.

Some properties of matrix exponential



- Because $[\hat{\omega}]$ is 3×3 skew-symmetric and $\hat{\omega}$ is of unit norm, we have

$$[\hat{\omega}]^3 = -[\hat{\omega}] \text{ and } [\hat{\omega}]^4 = -[\hat{\omega}]^2.$$

Recall that $\sin\theta = \theta - \frac{1}{3!}\theta^3 + \cdots$ and $\cos\theta = 1 - \frac{1}{2!}\theta^2 + \cdots$.

- We conclude from the 2 points above that

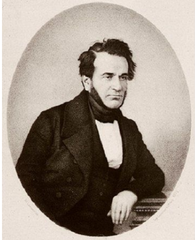
$$\begin{aligned} e^{[\hat{\omega}]\theta} &= I + [\hat{\omega}]\theta + \frac{1}{2!}[\hat{\omega}]^2 + \frac{1}{3!}[\hat{\omega}]^3 + \cdots \\ &= I + (\theta - \frac{\theta^3}{3!} + \cdots)[\hat{\omega}] + (\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \cdots)[\hat{\omega}]^2 \end{aligned}$$

Rodrigues formula

- We have thus shown the following, known as Rodrigues formula:

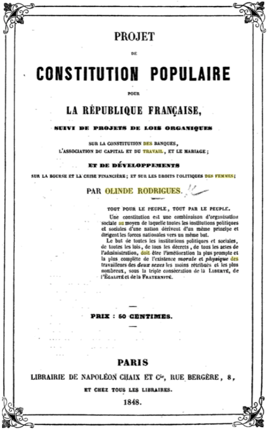
$$Rot(\hat{\omega}, \theta) = e^{[\hat{\omega}]\theta} = I + \sin\theta[\hat{\omega}] + (1 - \cos\theta)[\hat{\omega}]^2$$

- We say that $(\hat{\omega}, \theta)$ are the exponential coordinates of the rotation matrix R if $R = e^{\hat{\omega}\theta}$.



La République fondée sur la liberté, l'égalité, la fraternité, doit reconnaître désormais au travail des femmes autant et plus de droits que l'ancien régime n'en reconnut autrefois à leur oisiveté féodale

Benjamin Olinde RODRIGUES



PROJET

DE

CONSTITUTION POPULAIRE

POUR

LA RÉPUBLIQUE FRANÇAISE,

SUIVI DE PROJETS DE LOIS ORGANIQUES

SUR LA CONSTITUTION DES BANQUES,
L'ASSOCIATION DU CAPITAL ET DU TRAVAIL, ET LE MARIAGE;

ET DE DÉVELOPPEMENTS

SUR LA BOURSE ET LA CRISE FINANCIÈRE; ET SUR LES DROITS POLITIQUES DES FEMMES;

PAR OLINDE RODRIGUES.

TOUT POUR LE PEUPLE, TOUT PAR LE PEUPLE.

Une constitution est une combinaison d'organisation sociale au moyen de laquelle toutes les institutions politiques et sociales d'une nation dérivent d'un même principe et dirigent les forces nationales vers un même but.

Le but de toutes les institutions politiques et sociales, de toutes les lois, de tous les décrets, de tous les actes de l'administration, doit être l'amélioration la plus prompte et la plus complète de l'existence *morale* et *physique* des travailleurs des *deux sexes* les moins rétribués et les plus nombreux, sous la triple consécration de la LIBERTÉ, de l'ÉGALITÉ et de la FRATERNITÉ.

PRIX : 50 CENTIMES.

PARIS

LIBRAIRIE DE NAPOLÉON CHAIX ET C^{ie}, RUE BERGÈRE, 8,

ET CHEZ TOUS LES LIBRAIRES.

1848.

Matrix logarithm

- Given a rotation matrix R, in order to obtain its exponential coordinates, we need to take its so-called *logarithm*:

$$\begin{aligned} \exp : [\omega]\theta \in \mathfrak{so}(3) &\longrightarrow R \in SO(3) \\ \log : R \in SO(3) &\longrightarrow [\hat{\omega}]\theta \in \mathfrak{so}(3) \end{aligned}$$

- We can expand each entry in Rodrigues formula and obtain, with $c_\theta = \cos\theta$ and $s_\theta = \sin\theta$

$$\begin{bmatrix} c_\theta + \hat{\omega}_1^2(1-c_\theta) & \hat{\omega}_1\hat{\omega}_2(1-c_\theta) - \hat{\omega}_3 s_\theta & \hat{\omega}_1\hat{\omega}_3(1-c_\theta) + \hat{\omega}_2 s_\theta \\ \hat{\omega}_1\hat{\omega}_2(1-c_\theta) + \hat{\omega}_3 s_\theta & c_\theta + \hat{\omega}_2^2(1-c_\theta) & \hat{\omega}_2\hat{\omega}_3(1-c_\theta) - \hat{\omega}_1 s_\theta \\ \hat{\omega}_1\hat{\omega}_3(1-c_\theta) - \hat{\omega}_2 s_\theta & \hat{\omega}_2\hat{\omega}_3(1-c_\theta) + \hat{\omega}_1 s_\theta & c_\theta + \hat{\omega}_3^2(1-c_\theta) \end{bmatrix}$$

Matrix logarithm

- From previous equation for R, we see that

$$\operatorname{tr} R := r_{11} + r_{22} + r_{33} = 1 + 2\cos\theta \longrightarrow \text{solve for } \theta.$$

- Set R equal to the above matrix, and compute $R^\top - R$ to obtain:

$$\begin{aligned} r_{32} - r_{23} &= 2\hat{\omega}_1 \sin\theta \\ r_{13} - r_{31} &= 2\hat{\omega}_2 \sin\theta \\ r_{21} - r_{12} &= 2\hat{\omega}_3 \sin\theta \end{aligned}$$

- We can write the above as

$$[\hat{\omega}] = \frac{1}{2\sin\theta}(R - R^\top).$$

$\longrightarrow$ valid when $\sin\theta \neq 0$.

Matrix logarithm when $\sin\theta = 0$

- If $\theta = 2k\pi$, we have rotated by 360 degrees, and the rotation is, in the end, independent of $\hat{\omega}$. It is undefined in this case, or any $\hat{\omega}$ does the job.
- If $\theta = (2k+1)\pi$, then Rodrigues formula is

$$R = I + 2[\hat{\omega}]^2.$$

Based on this formula, we find

$$\hat{\omega}_i = \pm\sqrt{\frac{r_{ii}+1}{2}}$$
$$2\hat{\omega}_i\hat{\omega}_j = r_{ij}$$

for $i \neq j$.