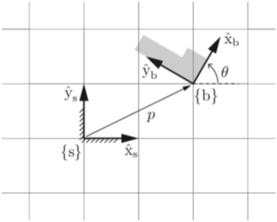
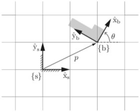
Introduction to Robotics
Lecture 3: Planar Rigid-Body Motions and 3D rotation matrices

Rigid-Body motions in the plane

- We now describe the position of a rigid body, first in 2D.
- Idea: attach a **frame** (=point+direction vectors) to the body. Each point in the body can be described by its position with respect to the frame.
  $\Rightarrow$ we only need to describe the position of the frame with respect to a reference frame.
- We need to choose one: body frame (b) / reference frame (s)



Rigid-Body motions in the plane

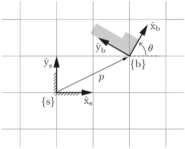


- x_s, y_s and s give the reference frame.
- The frame b is described by **three points**: p, x_b, y_b, and is attached to a rigid body. We consider p a point, but x_b, y_b to be vectors (think of x_b as $\vec{px_b}$)
- To describe the vectors x_b, y_b with respect to x_s, y_s, we need four numbers: indeed, we have

$$x_b = \cos\theta x_s + \sin\theta y_s$$
$$y_b = -\sin\theta x_s + \cos\theta y_s.$$

- Do not confuse a **vector** and its **coordinates**. The coordinates of a vector are defined with respect to a given basis; the same 'physical' vector has different coord. with respect to different bases.

Configuration and DoFs



- We put these in the matrix

$$P = P_{sb} = \begin{pmatrix} x_b & y_b \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

  Note that R only has one free parameter, θ. In fact,

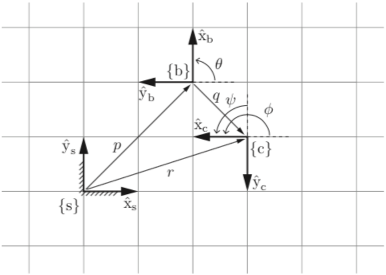
$$(*) \quad R^\top R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \longrightarrow 3 \text{ constraints on entries of } R$$

- R_{sb}: **the columns of R_{sb} are the vectors of b in basis s.**
- We call *any* R satisfying $(*)$ a **rotation or orthogonal matrix**.
- We can represent the position of the rigid body through the vector p and the rotation matrix R: frame $b = (\mathbf{R}, \mathbf{p})$

Change of coordinates and rotation matrices

- Consider a vector $v = v_{1s}x_s + v_{2s}y_s$, expressed in the reference frame s. To express it in the *body frame coordinates*, we need to solve
$$v = v_{1s}x_s + v_{2s}y_s = v_{1b}x_b + v_{2b}y_b$$
for v_{1b}, v_{2b}.
- We have $[x_b \ y_b] = [x_s \ y_s]R_{sb}$ and from the previous bullet, $[x_s \ y_s]\begin{pmatrix} v_{1s} \\ v_{2s} \end{pmatrix} = [x_b \ y_b]\begin{pmatrix} v_{1b} \\ v_{2b} \end{pmatrix}$. We conclude that
$$\begin{pmatrix} v_{1b} \\ v_{2b} \end{pmatrix} = R_{sb}^{-1}\begin{pmatrix} v_{1s} \\ v_{2s} \end{pmatrix}.$$
- Recall that R_{sb} is obtained by **writing the coordinate vectors of** basis b in basis s.

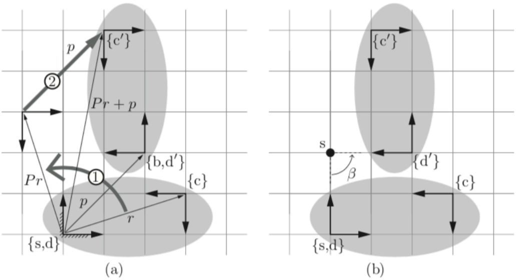
Changing frames



- Let (R_{sc}, r) denote the frame $\{c\}$ with respect to $\{s\}$ and (R_{sb}, p) the frame $\{b\}$ with respect to $\{s\}$. We can also represent $\{c\}$ with respect to $\{b\}$ as (R_{bc}, q) (note that this is with respect to x_b, y_b!). Then we have

$$R_{sc} = R_{sb}R_{bc}$$
$$r = R_{sb}q + p$$

Motions



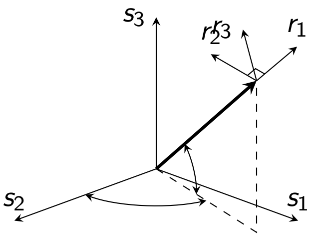
- Rigid body (ellipse) with frames c and d. The frame $d \equiv s$ at first, and $c = (R, r)$ in s.
- Then d move to d' (and $d' \equiv b$) with $b = (P, p)$ in s. This motion sends c to $c' = (R', r')$ with

$$R' = PR$$
$$r' = Pr + p$$

Special Euclidean Group $SE(2)$

- We can nicely summarize the above as follows: we say that a 3×3 matrix S is in the Euclidean Group of dimension 2 if it can be written as
$$S = \begin{pmatrix} R & r \\ \mathbf{0} & 1 \end{pmatrix}$$
where $R \in \mathbb{R}^{2\times 2}$ is a rotation matrix (i.e. $R^\top R = I$ and $\det R = 1$), and $r \in \mathbb{R}^2$.
- We can verify that
$$S_1 S_2 = \begin{pmatrix} R_1 R_2 & R_1 r_2 + r_1 \\ \mathbf{0} & 1 \end{pmatrix}$$
is again an element of $SE(2)$, which is exactly the composition of motions!
- Furthermore, S^{-1} corresponds to the inverse motion to the motion S.

Frames in 3D



- We write the coordinates of the frame R in the reference frame s as

$$r_i = r_{1i}s_1 + r_{2i}s_2 + r_{3i}s_3.$$

- Since the r_i are orthonormal, we have

$$\|r_i\|^2 = 1 = r_{1i}^2 + r_{2i}^2 + r_{3i}^2, \text{ and}$$

$$r_i^\top r_j = 0 \text{ for } i \neq j$$

Frames in 3D

- For example,

$$r_1^\top r_2 = (r_{11}s_1 + r_{21}s_2 + r_{31}s_3)^\top (r_{12}s_1 + r_{22}s_2 + r_{32}s_3) \\ = r_{11}r_{12} + r_{21}r_{22} + r_{31}r_{32}$$

  where we used fact that the s_i are orthonormal.

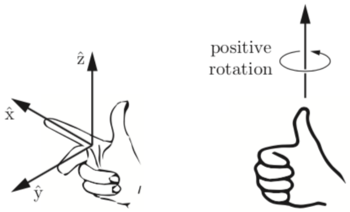
- Set $R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$. The above constraints are summarized as

$$R^\top R = I,$$

  where I is the 3×3 identity matrix.

- The matrix R is a **rotation or orthogonal** matrix in 3D.

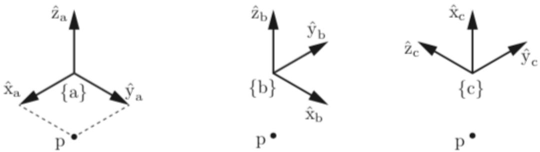
Right-handed frames



- A frame is right-handed, or positively oriented, if its x, y and z axes align with the index, middle finger and thumb respectively.
- Mathematically, a frame R is right-handed if the determinant of R is positive (in fact 1):

$$\det R = 1 \leftrightarrow \text{ frame is right-handed}$$

Rotation matrices in 3D



- Rotation matrices are also used to change frame: the point p expressed in the frames a, b and c has coordinates

$$p_a = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, p_b = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, p_c = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}.$$

- The matrices of the frames are

$$R_{sa} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, R_{sb} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, R_{sc} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Rotations matrices

- We used the convention R_{ab} for the matrix whose **columns are the coordinates of the frame b expressed in the frame a**. We have the composition law:

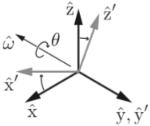
$$R_{ab}R_{bc} = R_{ac} \text{ and } R_{ab}^{-1} = R_{ba}.$$

- If p_a is the vector p expressed in frame a, then $R_{ba}p_a = p_b$.
- Why is it called a rotation matrix? Rotating a vector v around the axis x axis by angle θ is given by

$$Rot(x, \theta)v = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} v,$$

and rotations around y and z axis are obtained similarly.
- An arbitrary rotation can be obtained by *composing* these elementary rotations!

Representing rotations with rotation vector



- Given a vector $\omega = (\omega_1, \omega_2, \omega_3)$, a rotation of angle θ around ω is given by the matrix

$$\mathrm{Rot}(\hat{\omega}, \theta) =$$

$$\begin{bmatrix} c_\theta + \hat{\omega}_1^2(1 - c_\theta) & \hat{\omega}_1\hat{\omega}_2(1 - c_\theta) - \hat{\omega}_3 s_\theta & \hat{\omega}_1\hat{\omega}_3(1 - c_\theta) + \hat{\omega}_2 s_\theta \\ \hat{\omega}_1\hat{\omega}_2(1 - c_\theta) + \hat{\omega}_3 s_\theta & c_\theta + \hat{\omega}_2^2(1 - c_\theta) & \hat{\omega}_2\hat{\omega}_3(1 - c_\theta) - \hat{\omega}_1 s_\theta \\ \hat{\omega}_1\hat{\omega}_3(1 - c_\theta) - \hat{\omega}_2 s_\theta & \hat{\omega}_2\hat{\omega}_3(1 - c_\theta) + \hat{\omega}_1 s_\theta & c_\theta + \hat{\omega}_3^2(1 - c_\theta) \end{bmatrix},$$

Group of rotation matrices

Definition

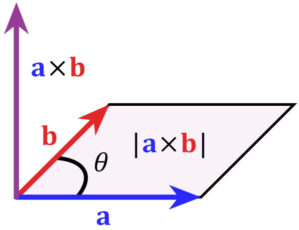
The *Special Orthogonal Group* $SO(3)$, or group of rotation matrices, is the set of all 3×3 matrices satisfying

$$R^\top R = I \text{ and } \det R = 1.$$

The set of matrices $SO(3)$ has the following properties: if $R_i \in SO(3)$, then

1. $R_1 R_2 \in SO(3)$: closure under multiplication
2. $R^{-1} = R^\top$
3. Rotation matrices preserve length of vectors: given $x \in \mathbb{R}^3$, and Rx, the rotated vector, we have $\|Rx\|^2 = x^\top R^\top R x = x^\top x$.
4. They do not commute in general: $R_1 R_2 \neq R_2 R_1$.
5. The inverse of a rotation matrix is a rotation matrix.

Cross-product



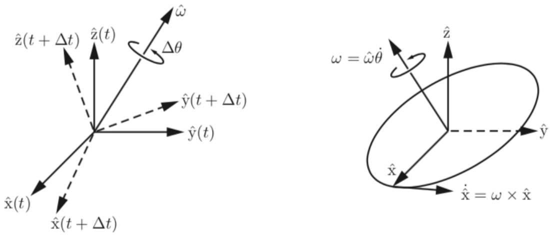
- Given a, b vectors in $\mathbb{R}^3$, their cross-product is given by

$$a \times b = \|a\| \|b\| \sin(\theta) N,$$

  where N is the unit normal vector to the plane containing a and b. The direction of N is given by the right-hand rule. The angle θ is the angle between a and b.
- In coordinates,

$$a \times b = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

Angular velocities



- Consider a small rotation of angle $\Delta\theta$ of the frame $\{x, y, z\}$ around the vector $\bar{\omega}$. It sends the frame vectors at time t to $x(t+\Delta t), \ldots, z(t+\Delta t)$ at time $t+\Delta t$.
- Observe that

$$x(t+\Delta t) \simeq x(t) + \bar{\omega}\Delta\theta \times x.$$

In the limit $\Delta t \to 0$, we have

$$\dot{x} = \omega \times x,$$

and similarly for y and z, where $\omega := \bar{\omega}\dot{\theta}$.

Angular velocities in reference frame

- Let $x = r_1, y = r_2$ and $z = r_3$ and set $R = [r_1 \;\; r_2 \;\; r_3]$. We can write
$$\dot{R} = [\omega \times r_1 \;\; \omega \times r_2 \;\; \omega \times r_3] = \omega \times R$$
- We want to get rid of the cross product and only have matrix multiplications.
- We need the following definition: to each vector $\omega \in \mathbb{R}^3$, we can associate uniquely a 3×3 matrix

$$\omega = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \leftrightarrow \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} := [\omega].$$

- Matrices as the one above are so that $A = -A^\top$. They are called *skew-symmetric*. The set of skew-symmetric matrices is denoted by $\mathfrak{so}(3)$

Angular velocities and change of frame

- We can now write $\dot{R} = \omega \times R$ as

$$\dot{R} = [\omega] R \Leftrightarrow [\omega] = \dot{R} R^{-1}.$$

Note: up to now, R, $\dot{R}$ and ω were expressed in an inertial reference frame.

- Recall that R describe the orientation of the body in the reference frame, i.e. $R_{sb} =: R$.
- We can express the rotation vector in the body frame as

$$\omega_b = R_{bs} \omega_s = R^{-1} \omega_s = R^\top \omega_s$$

where $\omega_s = \omega$. Recall that $R_{sb} = R_{bs}^{-1}$

A useful identity

Proposition

For $\omega \in \mathbb{R}^3$ *and* $R \in SO(3)$*, we have*

$$R[\omega]R^\top = [R\omega].$$

Proof. Let $r_i^\top$ be the ith row of R. Recall that if $M =$ is 3×3 matrix with columns $\{a, b, c\}$, then $\det M = a^\top(b \times c) = c^\top(a \times b) = b^\top(c \times a)$. We have

$$\begin{aligned} R[\omega]R^{\mathrm{T}} &= \begin{bmatrix} r_1^{\mathrm{T}}(\omega \times r_1) & r_1^{\mathrm{T}}(\omega \times r_2) & r_1^{\mathrm{T}}(\omega \times r_3) \\ r_2^{\mathrm{T}}(\omega \times r_1) & r_2^{\mathrm{T}}(\omega \times r_2) & r_2^{\mathrm{T}}(\omega \times r_3) \\ r_3^{\mathrm{T}}(\omega \times r_1) & r_3^{\mathrm{T}}(\omega \times r_2) & r_3^{\mathrm{T}}(\omega \times r_3) \end{bmatrix} \\ &= \begin{bmatrix} 0 & -r_3^{\mathrm{T}}\omega & r_2^{\mathrm{T}}\omega \\ r_3^{\mathrm{T}}\omega & 0 & -r_1^{\mathrm{T}}\omega \\ -r_2^{\mathrm{T}}\omega & r_1^{\mathrm{T}}\omega & 0 \end{bmatrix} \\ &= [R\omega], \end{aligned}$$

Angular velocities and change of frame

- Recall that $[\omega_s] = \dot{R}R^{-1}$.
- We have $\omega_b = R_{sb}^{\top}\omega_s \Leftrightarrow [\omega_b] = [R_{sb}^{\top}\omega_s]$.
- Using the previous Proposition, we have

$$[\omega_b] = R^{\top}[\omega_s]R = R^{\top}(\dot{R}R^{\top})R = R^{\top}\dot{R} = R^{-1}\dot{R}.$$

- We have in summary: for $R_{sb} = R$ the orientation of a rigid body in $3D$ with angular velocity vector ω_s (in reference frame) or ω_b (in body frame), we have

$$\dot{R}R^{-1} = [\omega_s]$$
$$R^{-1}\dot{R} = [\omega_b]$$