Introduction to Robotics
Lecture 2: Configuration Space Representation and holonomic/non-holonomic constraints

Configuration and DoFs

- The number of DoF's of a C-space gives us its dimension.
- Does it imply that if dof $= n$, then the C-space is $\mathbb{R}^n$? No: spaces of the same dimension can be *different* and are not necessarily $\mathbb{R}^n$. This impacts the way we set *coordinates* on the C-space.
- Two spaces are (topologically) *equivalent* is they can be continuously deformed into the other without cutting or gluing.

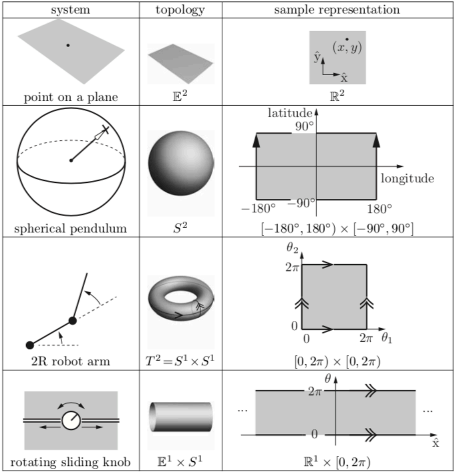
Examples

Configuration and DoFs

- The C-space of a point in the plane is $\mathbb{R}^2$.
- An *angle variable* is an element of the circle, denoted by S^1.
- Rotations in 3D are more complex. They are *not* given by three independent angles!
- C-space of rigid body in the plane is $\mathbb{R}^2 \times S^1$.
- C-space of PR-robot arm is $\mathbb{R} \times S^1$. C-space of 2R robot arm in $S^1 \times S^1 := T^2$.
- C-space of a planar rigid body with a 2R robot arm: $\mathbb{R}^2 \times S^1 \times T^2$.
- For S^2 the 2-sphere: $S^1 \times S^1 \neq S^2$!

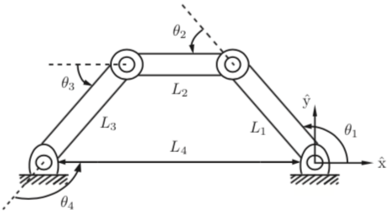
Configuration Space Representation

- A choice of *n free* coordinates to represent an *n*-dimensional space is called an *explicit* parametrization. They are *not unique* for a given C-space. For example, we can choose where to put the 'origin', or 'default configuration.



Configuration Space Representation

- An *implicit representation* is given by *constrained coordinates*. It is often easier to obtain than an explicit representation.



- Four angles + relations

$$L_1\cos(\theta_1) + L_2\cos(\theta_1+\theta_2) + \cdots + L_4\cos(\theta_1+\theta_2+\cdots+\theta_4) = 0$$
$$L_1\sin(\theta_1) + L_2\sin(\theta_1+\theta_2) + \cdots + L_4\sin(\theta_1+\theta_2+\cdots+\theta_4) = 0$$
$$\theta_1+\theta_2+\theta_3+\theta_4-2\pi = 0$$

$\Rightarrow$ 1 DoF.

Velocity constraints

- When the robot has mass/inertia, we need to include dynamical variable to describe its state. E.g., we need the position and momentum of a particle to describe its motion.
- Given a description of a state space with variables

$$(q_1, q_2, p_1, p_2), \text{ with } \dot{q}_i = p_i,$$

  constraints on the velocity can be either *holonomic* or *non-holonomic*: we explain the difference in the following slides.
- In a nutshell: holonomic constraints decrease the dimension of the C-space, non-holonomic constraints do not.

Velocity constraints

- Use $x \in \mathbb{R}^n$ for the position coordinates.
- We consider constraints given by

$$A(x)\dot{x} = 0,$$

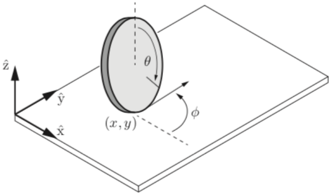
  where $A(x)$ is a matrix. These are called *Pfaffian constraints.*
- Given a set of Pfaffian constraints, we say they are *integrable* if we can find a function g such that $\frac{\partial g}{\partial x} = A$. Such constraints are *holonomic*.
- Why? If such g exists, the constraints $A(x)\dot{x} = 0$ are the same as the constraints on the position variables $g(x) = c$:

$$\frac{d}{dt}g(x(t)) = \frac{\partial g}{\partial x}\dot{x} = A(x)\dot{x} = 0.$$

- If no such g exists, the constraints are called *non-holonomic*.

Velocity constraints

- Consider a rolling disk on the plane. Described by (x, y, θ, ϕ).



- The coin rolls without slipping: its always goes in the direction $(\cos\phi, \sin\phi)$ with speed $r\dot{\theta}$:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = r\dot{\theta} \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}.$$

- Set $[q_1 \ q_1 \ \cdots \ q_4] = [x \ y \ \phi \ \theta]$, we have the Pfaffian constraints

$$\begin{pmatrix} 1 & 0 & 0 & -r\cos q_3 \\ 0 & 1 & 0 & -r\sin q_3 \end{pmatrix} \dot{q} = 0.$$

- These are not holonomic: intuitively, no decrease in dimension of C-space.

How to check for non-holonomy of constraints

- A **necessary** condition for non-holonomy: if there exists g such that $\frac{\partial g}{\partial x} = A$, then $A_{ij} = \frac{\partial g_i}{\partial x_j}$.
- Using equality of the mixed derivatives, we obtain the necessary condition:

$$\frac{\partial A_{ij}}{\partial x_k} = \frac{\partial A_{ik}}{\partial x_j}.$$

- Consider a rolling disk on the plane. Described by (x, y, θ, ϕ).
- These are not holonomic: mathematically, no g with $\frac{\partial g}{\partial q} = A$ exists: no equality of mixed derivatives!
- If such g exists and is twice differentiable, then $\frac{\partial^2 g}{\partial x \partial y} = \frac{\partial^2 g}{\partial y \partial x}$.
- Here, $\frac{\partial g_1}{\partial q_4} = -r \cos q_3$ and thus $\frac{\partial^2 g_1}{\partial q_4 \partial q_3} = r \sin q_3$.
- Similarly, $\frac{\partial g_1}{\partial q_3} = 0$ and thus $\frac{\partial^2 g_1}{\partial q_3 \partial q_4} = 0 \Rightarrow$ contradiction.