

Grubler's Formula

$$Dof = m(N - J - 1) + \sum_i f_i$$

N: number of links

J: number of joints

m: Dof of rigid body, if planar m=3, if spatial m=6

f_i : number of freedom from joint i

2D Rotation matrix

$$R(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

3D Rotation matrix

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3×3 skew-symmetric matrix representation

$$[x] = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$$

Angular velocity in the fixed frame and body frame

$$[\omega_s] = \dot{R}R^T$$

$$[\omega_b] = R^T \dot{R}$$

Rotation matrix in exponential coordinate representation

$$R(\hat{\omega}, \theta) = e^{[\hat{\omega}]\theta} = I + \sin \theta [\hat{\omega}] + (1 - \cos \theta) [\hat{\omega}]^2$$

Matrix Logarithm Method

$$\text{tr}(R) = r_{11} + r_{22} + r_{33} = 1 + 2\cos \theta$$

$$[\hat{\omega}] = \frac{1}{2\sin \theta} (R^T - R)$$

The inverse of the transformation matrix

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \quad T^{-1} = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix}$$

Twist

$$\mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix} \quad [\mathcal{V}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix}$$

Twist in fixed and body frame

$$[\mathcal{V}_b] = T^{-1} \dot{T} \quad [\mathcal{V}_s] = \dot{T} T^{-1}$$

$$\mathcal{V}_s = [Ad_T] \mathcal{V}_b \quad [Ad_T] = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix}$$

Screw

$$\text{screw axis: } \mathcal{S} = \{q, \hat{s}, h\}, \quad \text{velocity: } \dot{\theta}$$

Screw to twist

$$\mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix} = \begin{bmatrix} \hat{s} \dot{\theta} \\ h \hat{s} \dot{\theta} - \hat{s} \dot{\theta} \times q \end{bmatrix}$$

Twist to screw

$$\hat{s} = \frac{\omega}{\|\omega\|} = \hat{\omega}, \quad \dot{\theta} = \|\omega\|, \quad h = \frac{\omega^T v}{\|\omega\|^2}$$

$$\text{if } \omega \neq 0, \quad q = \frac{\hat{s} \times v}{\dot{\theta}}$$

Screw axis

$$\|\omega\| = 1 : v = -\omega \times q + h\omega$$

$$\|v\| = 1, \omega = 0 : \text{translate along } v$$

Exponential Coordinates for Rigid Body Motions(S $\rightarrow$ T)

$$T = e^{[S]\theta} = \begin{bmatrix} e^{[\omega]\theta} & G(\theta)v \\ 0 & 1 \end{bmatrix}$$

$$G(\theta) = I\theta + (1 - \cos\theta)[w] + (\theta - \sin\theta)[w]^2$$

Logarithm of rigid body motions(T $\rightarrow$ S)

$$tr(R) = r_{11} + r_{22} + r_{33} = 1 + 2\cos\theta$$

$$[\hat{\omega}] = \frac{1}{2\sin\theta}(R - R^T)$$

$$v = G^{-1}(\theta)p \quad G^{-1}(\theta) = \frac{1}{\theta}I - \frac{1}{2}[w] + \left(\frac{1}{\theta} - \frac{1}{2}\cot\left(\frac{\theta}{2}\right)\right)[w]^2$$

Forward Kinematics

$$\text{Screw in fixed frame: } T(\theta) = e^{[S_1]\theta_1} e^{[S_2]\theta_2} \dots e^{[S_n]\theta_n} M$$

$$\text{Screw in body frame: } T(\theta) = M e^{[B_1]\theta_1} e^{[B_2]\theta_2} \dots e^{[B_n]\theta_n}$$

Velocity Kinematics

Space Jacobian

$$\mathcal{V}_s = J_s(\theta) \dot{\theta} = \begin{bmatrix} J_{s,1} & J_{s,2} & \dots & J_{s,n} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \vdots \\ \dot{\theta}_n \end{bmatrix}$$

$$J_{s,1} = S_1 \quad J_{s,2} = Ad_{e^{[S_1]\theta_1}}(S_2) \quad \dots \quad J_{s,n} = Ad_{e^{[S_1]\theta_1} \dots e^{[S_{n-1}]\theta_{n-1}}}(S_n)$$

Body Jacobian

$$\mathcal{V}_b = J_b(\theta) \dot{\theta} = \begin{bmatrix} J_{b,1} & J_{b,2} & \dots & J_{b,n} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \vdots \\ \dot{\theta}_n \end{bmatrix}$$

$$J_{b,n} = \mathcal{B}_n \quad J_{b,n-1} = Ad_{e^{-[\mathcal{B}_n]\theta_n}}(\mathcal{B}_{n-1}) \quad \dots \quad J_{b,1} = Ad_{e^{-[\mathcal{B}_n]\theta_n} \dots e^{-[\mathcal{B}_2]\theta_2}}(\mathcal{B}_1)$$

Relationship between Space and Body Jacobian

$$[\mathcal{V}_s] = \dot{T}_{sb} T_{sb}^{-1} \quad [\mathcal{V}_b] = T_{sb}^{-1} \dot{T}_{sb}$$

$$\mathcal{V}_s = Ad_{T_{sb}}(\mathcal{V}_b) \quad \mathcal{V}_b = Ad_{T_{bs}}(\mathcal{V}_s)$$

$$J_s(\theta) = Ad_{T_{sb}}(J_b(\theta)) = [Ad_{T_{sb}}] J_b(\theta)$$

$$J_b(\theta) = Ad_{T_{bs}}(J_s(\theta)) = [Ad_{T_{bs}}] J_s(\theta)$$

Cross Product

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = i \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} - j \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} + k \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix}$$