Lecture 13:
inverse kinematics I

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Leonardo Da Vinci’s Robot

- Leonardo's robot, or Leonardo's mechanical knight, was a humanoid automaton designed and possibly constructed by Leonardo da Vinci around 1495
- The robot knight could stand, sit, raise its visor and independently maneuver its arms, and had an anatomically correct jaw, all operated by a series of pulleys and cables
    - The robot has been built faithfully based on Leonardo's design, and was found to be fully functional
- **Question:** is this a robot?

A few geometric tricks

Law of Cosines

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Two argument arctan function atan2

$$\text{atan2}(y,x) = \begin{cases} \text{atan}\left(\frac{y}{x}\right) \text{ if } x > 0 \\ \text{atan}\left(\frac{y}{x}\right) + \pi \text{ if } x < 0 \text{ and } y \geq 0 \\ \text{atan}\left(\frac{y}{x}\right) - \pi \text{ if } x < 0 \text{ and } y < 0 \\ \frac{\pi}{2} \text{ if } x = 0 \text{ and } y > 0 \\ \frac{-\pi}{2} \text{ if } x = 0 \text{ and } y < 0 \\ \text{undefined if } x = 0 \text{ and } y = 0 \end{cases}$$

Euler Angles

Given some arbitrary rotation, how to find the associated angles?

- ZYX Euler angles represent a rotation as follows:

Euler Angles (see Appendix B)

$$R(\alpha,\beta,\gamma) = \begin{bmatrix} c_\alpha c_\beta & c_\alpha s_\beta s_\gamma - s_\alpha c_\gamma & c_\alpha s_\beta s_\gamma + s_\alpha s_\gamma \\ s_\alpha c_\beta & s_\alpha s_\beta s_\gamma + c_\alpha c_\gamma & s_\alpha s_\beta c_\gamma - c_\alpha s_\gamma \\ -s_\beta & c_\beta s_\gamma & c_\beta c_\gamma \end{bmatrix}$$

If r_{ij} denotes the ij^{th} element in R, then

1. If $r_{31} \neq \pm 1$, set $\beta = \text{atan2}\left(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}\right), \alpha = \text{atan2}(r_{21}, r_{11}), \gamma = \text{atan2}(r_{32}, r_{33})$
2. If $r_{31} = -1$, then $\beta = \frac{\pi}{2}$ and there are infinite solutions for α and γ. One solution is: $\alpha = 0,\ \gamma = \text{atan2}(r_{12}, r_{22})$
3. If $r_{31} = 1$, then $\beta = -\frac{\pi}{2}$ and there are infinite solutions for α and γ. One solution is: $\alpha = 0,\ \gamma = -\text{atan2}(r_{12}, r_{22})$

What is going on in this class?

- Rigid Body Motion
- Forward Kinematics
  - Calculate the position of the end-effector of an open chain given joint angles
    $$x(t) = f\big(\theta(t)\big)$$
- Velocity Kinematics
  - Calculate the velocity (twist!) of the end-effector
    $$\frac{d}{dt}x(t) = \frac{\partial f}{\partial \theta}(\theta) \cdot \dot{\theta}$$
- Inverse Kinematics
  - Computes the possible joint angles from the pose of the end-effector
  - Given $T(\theta)$, find solutions θ that satisfy $T(\theta) = X$

Inverse Kinematics in Animation

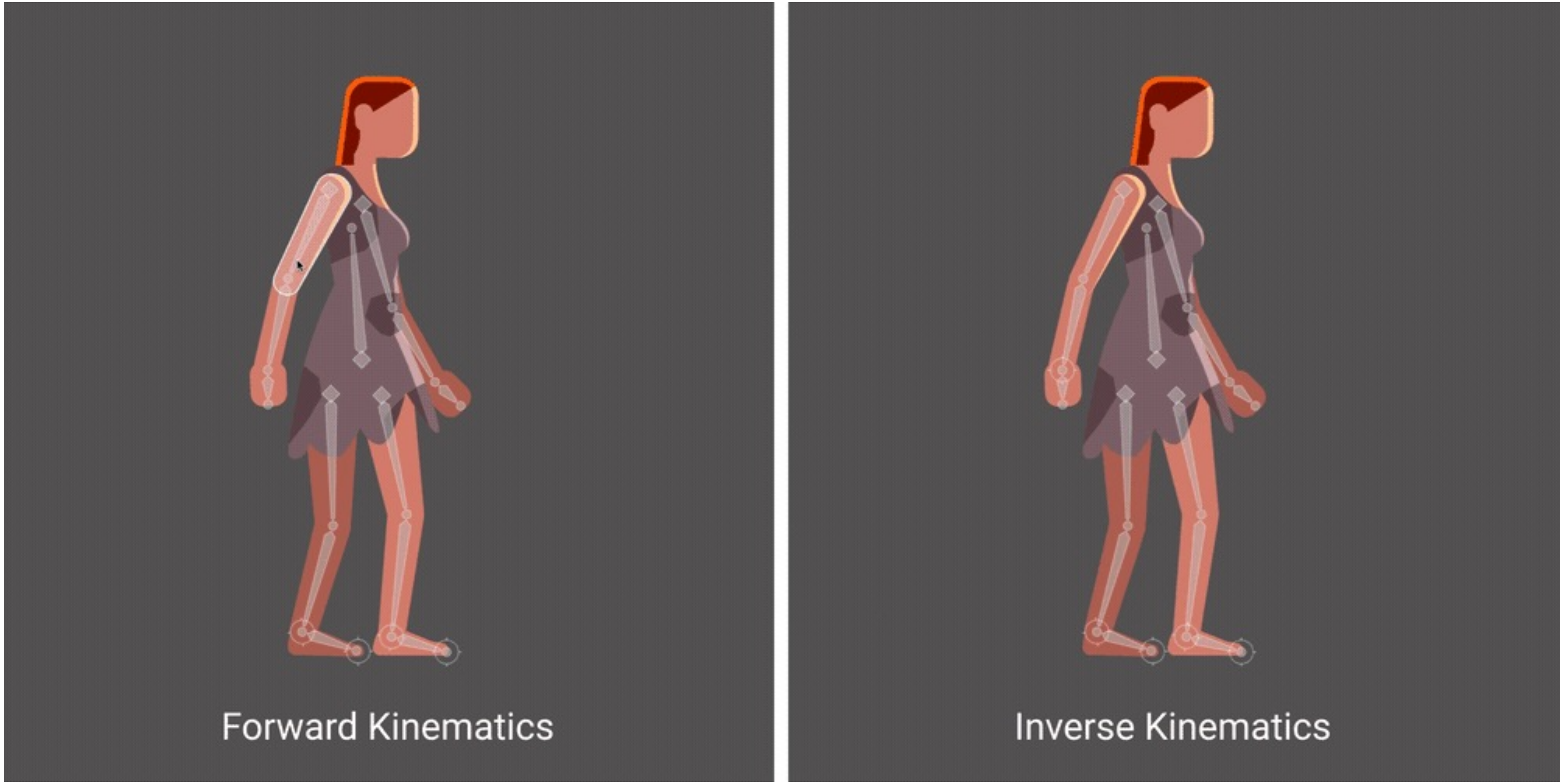


Image Credit: Rive

Inverse Kinematics

- **Forward Kinematics:** computes the end-effector position from joint angles:

- **Inverse Kinematics:** computes the possible joint angles from the pose of the end-effector

- Given $T(\theta)$, find solutions θ that satisfy $T(\theta) = X$

Inverse Kinematics (IK)

Analytic Inverse Kinematics

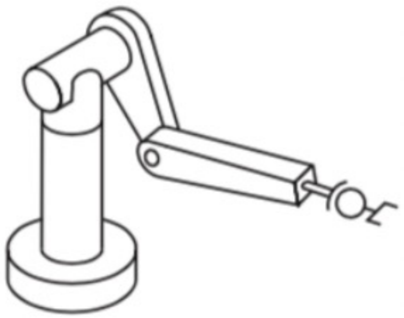
Analytic inverse kinematics

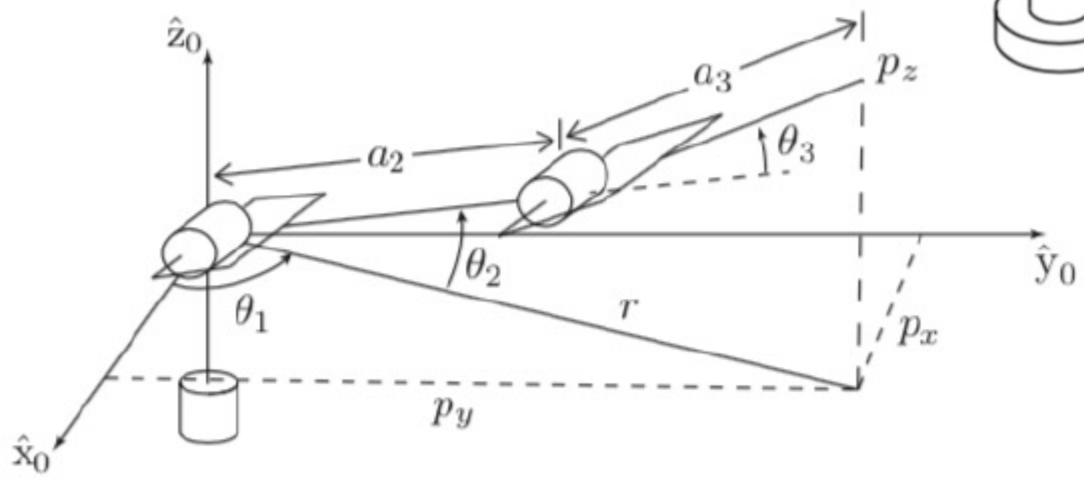
- In the 2R example, there were 2 DOFs for the end-effector and 2 joint angles. This implied that there was a **finite number of solutions.**
- If there are more joint angles than DOFs of the end-effector, there may be **an infinite number of solutions.**
- In practice, we will assume that #DOFs of end-effector = #joint angles.
- For now, assume in general:

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} \dots e^{[\mathcal{S}_6]\theta_6} M$$

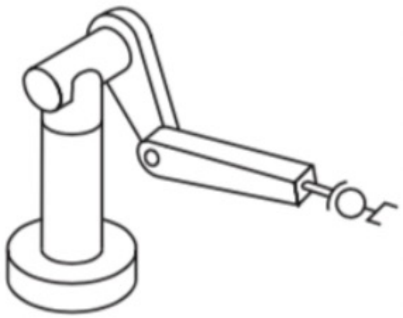
and that we are given an end-effector pose $X \in SE(3)$.

6R PUMA arm





6R PUMA arm – orientation



Numerical Inverse Kinematics

- **Forward Kinematics:** $\theta \rightarrow \mathrm{T}(\theta)$

- **Inverse Kinematics:** $X \rightarrow \theta$

- When the analytic solution is hard or impossible to come by, we **numerically** solve:

Summary

- Introduced analytic **inverse kinematics**, which helps us find the joint angles that will produce a desired end-effector pose
  - We can do this with the help of **atan2** and **ZYX Euler Angles**
  - However, this is only works for relatively simple robot chains