lecture 10
velocity kinematics II

Prof. DC
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Modern Robotics Ch. 5.1-5.3
Admin

• HW5 due on Friday (3/3)
• First guest lecture will be on Thursday (3/9)
  • Attendance is required
  • You will submit a short reflection after
• This week’s homework party is in 1232 CSL Studio
Extra Credit Opportunity

• Submit an informational/tutorial video for some course topic
• The video should be ~5 minutes in length. You will be graded on the following:
  – How challenging is the topic?
  – Did you provide an interesting motivating example?
  – How well did you explain the topic?
  – Did you provide an informative worked example?
  – How effective are the visualizations used?
• Check out examples on the website, and/or 3Blue1Brown or the Kahn Academy on YouTube
Refresher on Twists

• What is a twist?
  • Spatial velocity: $\mathcal{V}_s = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix}$, $[\mathcal{V}_s] = \begin{bmatrix} [\omega_s] \\ v_s \\ 0 \end{bmatrix} = \dot{T}T^{-1}$

• What is the adjoint map?
  • Given $T = (R, p)$, $[\text{Ad}_T] = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \in \mathbb{R}^{6 \times 6}$
  • $\mathcal{V}' = [\text{Ad}_T] \mathcal{V} = \text{Ad}_T(\mathcal{V})$
  • $[\mathcal{V}'] = T[\mathcal{V}]T^{-1}$
Velocity Kinematics

**Forward Kinematics**
Calculate the position of the end-effector of an open chain given joint angles

\[ x(t) = f(\theta(t)) \]

**Velocity Kinematics**
Calculate the velocity (twist!) of the end-effector frame

\[ \frac{d}{dt} x(t) = \frac{\partial f}{\partial \theta}(\theta) \cdot \dot{\theta} = J(\theta) \cdot \dot{\theta} \]
Jacobian and Manipulability

• Suppose joint limits are \( \dot{\theta}_1^2 + \dot{\theta}_2^2 \leq 1 \)
• The ellipsoid obtained by mapping through the Jacobian is called the manipulateability ellipsoid

Configurations for which the end-effector cannot move in one or more directions instantaneously is a kinematic singularity. 
\( J(\theta) \) is not full rank, <6 linearly independent columns.
Computing the Jacobian

• Recall forward kinematics:
  \[
  T(\theta) = e^{[S_1]\theta_1}e^{[S_2]\theta_2} \cdots e^{[S_n]\theta_n}M
  \]

• Take the derivative:
  \[
  \dot{T}(\theta) = \frac{d}{dt} \left( e^{[S_1]\theta_1}e^{[S_2]\theta_2} \cdots e^{[S_n]\theta_n}M + \cdots + e^{[S_1]\theta_1}e^{[S_2]\theta_2} \cdots \frac{d}{dt} (e^{[S_n]\theta_n})M \right)
  = [S_1] \dot{\theta}_1 e^{[S_1]\theta_1} \cdots e^{[S_n]\theta_n}M + \cdots + e^{[S_1]\theta_1} \cdots [S_n] \dot{\theta}_n e^{[S_n]\theta_n}M
  \]

• Take the inverse:
  \[
  T^{-1}(\theta) = M^{-1} e^{-[S_n]\theta_n} \cdots e^{-[S_1]\theta_1}
  \]

• Recall: \([V_S] = \dot{T} T^{-1}\)
  \[
  \dot{T}T^{-1} = [S_1] \dot{\theta}_1 + e^{[S_1]\theta_1} [S_2] e^{-[S_1]\theta_1} \dot{\theta}_2 + e^{[S_1]\theta_1} e^{[S_2]\theta_2} [S_3] e^{-[S_2]\theta_2} e^{-[S_1]\theta_1} \dot{\theta}_3 + \cdots
  \]

  \[\mathbf{J}(\theta) \cdot \dot{\theta}\]

\[\begin{align*}
\mathbf{J}(\theta) &= \begin{bmatrix}
[S_1] \dot{\theta}_1 + e^{[S_1]\theta_1} [S_2] e^{-[S_1]\theta_1} \dot{\theta}_2 + e^{[S_1]\theta_1} e^{[S_2]\theta_2} [S_3] e^{-[S_2]\theta_2} e^{-[S_1]\theta_1} \dot{\theta}_3 + \cdots
\end{bmatrix}
\end{align*}\]
Computing the Jacobian (2)

• Expressed via Adjoint:
  \[ \mathcal{V}_s = S_1 \dot{\theta}_1 + Ad_{e[S_1,\theta_1]} (S_2) \dot{\theta}_2 + Ad_{e[S_1,\theta_1,\theta_2]} (S_3) \dot{\theta}_3 + \ldots \]
  \[ \mathbf{J}_{s_1} \quad \mathbf{J}_{s_2} \quad \mathbf{J}_{s_3} \ldots \mathbf{J}_{s_n} \]

• Gives our Space Jacobian: \( \mathcal{V}_s = J_s(\theta) \dot{\theta} \), \( J_s(\theta) = [\mathbf{J}_{s_1} \; \mathbf{J}_{s_2} \ldots \; \mathbf{J}_{s_n}] \)

we have: \( \mathbf{J}_{s_1} = S_1 \), \( \mathbf{J}_{s_i} = Ad_{e[S_1,\ldots,S_{i-1},\theta_i]} (S_i) \)

• **Intuition:** If \( T_{i,M} \) is the configuration when you set joints \( \theta_1, \ldots, \theta_i \) and leave remaining joints at \( 0 \), then \( J_{s_i} \) is the screw vector at joint \( i \), for an arbitrary \( \theta \)
Visualizing the Jacobian

- By inspection, let’s find $J_{s3}$
- Ignore joints $\theta_3, \theta_4, \theta_5$, as they do not displace axis 3 relative to \{s\}
- If $\theta_1 = 0$ and $\theta_2$ is arbitrary, then $T_{ss'} = e^{[S_2]_\theta_2}$
- If $\theta_1$ is arbitrary, then $T_{ss''} = e^{[S_1]_\theta_1} e^{[S_2]_\theta_2}$
- $J_{s3} = [Ad_{T_{ss''}}]S_3 = [Ad_{e^{[S_1]_\theta_1} e^{[S_2]_\theta_2}}]S_3$
Methods for Computing the Jacobian

By Inspection

\[ J_{si} = (\omega_{si}, \nu_{si}) \] where \( Ad_{T_{i-1}} \) is implicit

By Definition

The space Jacobian \( J_s(\theta) \in \mathbb{R}^{6 \times n} \) relates the joint rate vector \( \dot{\theta} \in \mathbb{R}^n \) to the spatial twist \( \nu_s \) via

\[ \nu_s = J_s(\theta) \dot{\theta}. \] (5.10)

The \( i \)th column of \( J_s(\theta) \) is

\[ J_{si}(\theta) = Ad_{q_{s_i}q_{s_{i-1}}^{-1}}(S_i), \] (5.11)

for \( i = 2, \ldots, n \), with the first column \( J_{s1} = S_1 \).
Example 1

1. $J_{s1} = S_1$, $w_{s1} = [0, 0, 1]^T$, $v_{s1} = [0, 0, 0]^T$

2. $w_{s2} = [0, 0, 1]^T$, $q_2 = [L_1 \cos \theta_1, L_1 \sin \theta_1, 0]^T$
   
   $v_{s2} = -w_{s2} \times q_2 = [L_1 \sin \theta_1, -L_1 \cos \theta_1, 0]^T$

3. $w_{s3} = [0, 0, 1]^T$
   
   $q_3 = [L_1 \cos \theta_1 + L_2 \cos (\theta_1 + \theta_2), L_1 \sin \theta_1 + L_2 \sin (\theta_1 + \theta_2), L_2]^T$

4. $w_{s4} = [0, 0, 0]^T$, $v_{s4} = [0, 0, 1]^T$

$J_s(\Theta) = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 \\
0 & L_1 S_1 & L_1 S_1 + L_2 S_{12} & 0 \\
0 & -L_1 C_1 & -L_1 C_1 - L_2 C_{12} & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}$
Example 2

What is $S_1$?

$$\omega_{sr} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad q_{s1} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$$v_{s1} = -\omega_{s1} \times q_{s1} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$$

$$J_{s1} = S,$$

$$y = J_{s1} \cdot \dot{\theta}$$
Example 3

1. $J_{s1} = S_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$

2. $\omega_{s2} = [0, -1, 0]^{\top}$
   $q_2 = [-2, 0, -\theta_1]^{\top}$
   $v_{s2} = [-\theta_1, 0, 2]^{\top}$
   $J_{s2} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}$

3. $\omega_{s3} = [0, -1, 0]^{\top}$
   $q_3 = \begin{bmatrix} -2 + 2\sin\theta_2 \\ 0 \\ -\theta_1 - 2\cos\theta_2 \end{bmatrix}$
   $v_{s3} = \begin{bmatrix} -\theta_1 - 2\cos\theta_2 \\ 0 \\ 2 - 2\sin\theta_2 \end{bmatrix}$
   $J_{s3} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ \theta_2 \end{bmatrix}$
Example 4

1. \( \omega_{s1} = [0, 0, 0] \) \( T \), \( q_{s1} = [-2, 2, 0] \)
2. \( \omega_{s2} = [-1, 0, 0] \) \( T \), \( q_{s2} = [-2, 2, 0] \)
3. \( \omega_{s3} = [0, 0, 0] \) \( T \), \( q_{s3} = [0, 1, 0] \)

Now compute: \( \ell \) \( \text{C}_{s1}\theta_1 \), \( \ell \) \( \text{C}_{s2}\theta_2 \), \( \ell \) \( \text{C}_{s3}\theta_3 \)
Body Jacobians

Our space Jacobian is $[\mathcal{V}_s] = \dot{T} T^{-1}$, and our body Jacobian is $[\mathcal{V}_b] = T^{-1} \dot{T}$

$$T(\theta) = e^{[S_1] \theta_1} e^{[S_2] \theta_2} \cdots e^{[S_n] \theta_n} M = Me^{[B_1] \theta_1} e^{[B_2] \theta_2} \cdots e^{[B_n] \theta_n}$$

$$\dot{T}(\theta) = M [B_1] \dot{\theta}_1 e^{[B_2] \theta_2} \cdots e^{[B_n] \theta_n} + \cdots + Me^{[B_1] \theta_1} [B_n] \dot{\theta}_n e^{[B_n] \theta_n}$$

$$T^{-1}(\theta) = e^{-[B_n] \theta_n} \cdots e^{-[B_1] \theta_1} M^{-1}$$

$$T^{-1} \dot{T}$$

$$= [B_n] \dot{\theta}_n + e^{-[B_n] \theta_n} [B_{n-1}] e^{[B_n] \theta_n} \dot{\theta}_{n-1}$$

$$+ e^{-[B_n] \theta_n} e^{-[B_{n-1}] \theta_{n-1}} [B_{n-2}] e^{[B_{n-1}] \theta_{n-1}} e^{[B_n] \theta_n} \dot{\theta}_{n-1}$$

$$+ e^{-[B_n] \theta_n} \cdots e^{-[B_2] \theta_2} [B_1] e^{[B_2] \theta_2} \cdots e^{[B_n] \theta_n} \dot{\theta}_1$$
The relationship between the Space and Body Jacobian

Recall that:

\[
\begin{align*}
[\mathcal{V}_s] &= \dot{T}_{sb} T_{sb}^{-1} \\
[\mathcal{V}_b] &= T_{sb}^{-1} \dot{T}_{sb} \\
\mathcal{V}_s &= \text{Ad}_{T_{sb}}(\mathcal{V}_b) \\
\mathcal{V}_b &= \text{Ad}_{T_{bs}}(\mathcal{V}_s) \\
\mathcal{V}_s &= J_s(\theta) \dot{\theta} \\
\mathcal{V}_b &= J_b(\theta) \dot{\theta}
\end{align*}
\]

\[
\text{Ad}_{T_{sb}}(\mathcal{V}_b) = J_s(\theta) \dot{\theta}
\]

• Apply \([\text{Ad}_{T_{bs}}]\) to both sides and recall that \([\text{Ad}_{TX}][\text{Ad}_Y] = [\text{Ad}_{XY}]:\)

\[
\begin{align*}
\text{Ad}_{T_{bs}}(\text{Ad}_{T_{sb}}(\mathcal{V}_b)) &= \mathcal{V}_b = \text{Ad}_{T_{bs}}(J_s(\theta) \dot{\theta}) \\
J_b(\theta) \dot{\theta} &= \text{Ad}_{T_{bs}}(J_s(\theta) \dot{\theta})
\end{align*}
\]

• Since this holds for all \(\dot{\theta}:\)

\[
\begin{align*}
J_b(\theta) &= \text{Ad}_{T_{bs}}(J_s(\theta)) = [\text{Ad}_{T_{bs}}] J_s(\theta) \\
J_s(\theta) &= \text{Ad}_{T_{sb}}(J_b(\theta)) = [\text{Ad}_{T_{sb}}] J_b(\theta)
\end{align*}
\]
Summary

• Defined velocity kinematics as means to compute the twist at the end-effector frame

• Interpreted the end-effector twist as a way to map actuator limits to possible end-effector velocities (directions of motion) and insight as to how close a pose is to a singularity

• Learned two methods for computing the Jacobian: (1) by inspection and (2) by using equations for defining the spatial and body twists

• Now that we can describe the motion of our robot, we’ll be moving on to planning, which allows us to tell our robot how to move