lecture 9
quick recap and
velocity kinematics I

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Modern Robotics Ch. 5.1-5.3
Admin

• HW4 due date moved to after exam 1
• Exam 1 next week (Mon – Wed)
  • Remember to signup for the exam through the CBTF website
  • Practice exam available on prairielearn
  • I recommend going through past homework
  • Review lecture on Thursday
  • No lecture on Tuesday 2/21
Who is Carl Gustav Jacob Jacobi?

• A German mathematician who contributed to elliptic functions, dynamics, differential equations, determinants, and number theory

• In Germany during the Revolution of 1848, Jacobi was politically involved with the Liberal club, which after the revolution ended led to his royal grant being temporarily cut off

• His PhD student, Otto Hesse, is known for the Hessian Matrix (second-order partial derivatives of a scalar-valued function, or scalar field)
Quick Course Recap

• We use our knowledge of the links, joints, and actuators to represent the configuration of the robot

• The number of degrees of freedom of the robot is the smallest number of coordinates needed to represent the configuration
Quick Course Recap

• We use our knowledge of the links, joints, and actuators to represent the **configuration** of the robot

• The number of **degrees of freedom** of the robot is the smallest number of coordinates needed to represent the configuration

• The **task space** is the space in which the robot’s task can be naturally expressed

• The **workspace** is the specification of the configurations that the end-effector of the robot can reach
Rotations and Transformations

• What do these matrices represent and what operations can they do?
Rotations and Transformations

• What do these matrices represent and what operations can they do?

• Exponential coordinates allow us to parameterize rotations by the rotation axis and angle of rotation and homogeneous transformations by the screw axis
Velocity Kinematics

**Forward Kinematics**
Calculate the position of the end-effector of an open chain given joint angles

\[
x(t) = f(\Theta(t))
\]

**Velocity Kinematics**
Calculate the velocity (twist!) of the end-effector frame

\[
\dot{x} = \frac{d}{dt} x(t) = \frac{\partial f(\Theta(t))}{\partial \Theta} \dot{\Theta}
\]

\[
J(\Theta) \rightarrow \text{jacobian of } f
\]
Example

FK: \[ x_1 = L_1 \cos \theta_1 + L_2 \cos (\theta_1 + \theta_2) \]
\[ x_2 = L_1 \sin \theta_1 + L_2 \sin (\theta_1 + \theta_2) \]

Differentiate:
\[ \dot{x}_1 = -L_1 \dot{\theta}_1 \sin \theta_1 - L_2 (\dot{\theta}_1 + \dot{\theta}_2) \sin (\theta_1 + \theta_2) \]
\[ \dot{x}_2 = L_1 \dot{\theta}_1 \cos \theta_1 + L_2 (\dot{\theta}_1 + \dot{\theta}_2) \cos (\theta_1 + \theta_2) \]

rearrange:
\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
-J_1(\theta) & -L_2 \sin(\theta_1 + \theta_2) \\
L_1 \cos \theta_1 + L_2 \cos(\theta_1 + \theta_2) & J_2(\theta)
\end{bmatrix}
\begin{bmatrix}
\dot{\theta}_1 \\
\dot{\theta}_2
\end{bmatrix}
\]

\[ \ddot{x} = J_1(\theta) \dot{\theta}_1 + J_2(\theta) \dot{\theta}_2 \]
Jacobian Mapping

If \( L_1 = L_2 = 1, \; \theta_1 = 0, \; \theta_2 = \frac{\pi}{4}, \) then:

\[
J \left( \begin{bmatrix} 0 \\ \frac{\pi}{4} \end{bmatrix} \right) = \begin{bmatrix} -0.71 & -0.71 \\ 1.71 & 0.71 \end{bmatrix}
\]

Assume actuator limits: \( \theta_i \in [-1, 1] \)

\[
J(\theta)
\]

limits \( \theta_2 \)

\( \theta_1 \rightarrow \)

\( \text{tool vel} \)

\( \dot{x}_1, \dot{x}_2 \)

\( A = (1, 1) \)

\( B, C, D \)
Jacobian and Manipulability

- Suppose joint limits are \( \dot{\theta}_1^2 + \dot{\theta}_2^2 \leq 1 \)
- The ellipsoid obtained by mapping through the Jacobian is called the **manipulability ellipsoid**
Jacobian and Statics

• Assume a force \( f_{\text{tip}} \) is applied on the end-effector. What torque (\( \tau \)) must be applied at the joint to keep the fixed position?
Jacobian and Statics

• Assume a force $f_{tip}$ is applied on the end-effector. What torque ($\tau$) must be applied at the joint to keep the fixed position?

• By conservation of power:
  
  $$f_{tip}^T v_{tip} = \tau^T \dot{\theta}, \quad \forall \dot{\theta}$$

• Since $v_{tip} = J(\theta)\dot{\theta}$, we have
  
  $$f_{tip}^T J(\theta)\dot{\theta} = \tau^T \dot{\theta}, \quad \forall \dot{\theta}$$

• Which gives:
  
  $$\tau = J^T(\theta)f_{tip}$$
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• Which gives:

$$\tau = J^T(\theta) f_{tip}$$

• If $J^T(\theta)$ is invertible, then given the limits of the torques at the joints, we can compute all the forces that can be counteracted at the end-effector
Computing the Jacobian (1)

• Recall forward kinematics:

\[ T(\theta) = e^{[S_1]^{\theta_1}}e^{[S_2]^{\theta_2}} \cdots e^{[S_n]^{\theta_n}}M \]
Computing the Jacobian (1)

- Recall forward kinematics:
  \[ T(\theta) = e^{[S_1] \theta_1} e^{[S_2] \theta_2} \cdots e^{[S_n] \theta_n} M \]

- Take the derivative:
  \[ \dot{T}(\theta) = \frac{d}{dt} (e^{[S_1] \theta_1} e^{[S_2] \theta_2} \cdots e^{[S_n] \theta_n} M + \cdots + e^{[S_1] \theta_1} e^{[S_2] \theta_2} \cdots \frac{d}{dt} (e^{[S_n] \theta_n}) M) \]
  \[ = [S_1] \dot{\theta}_1 e^{[S_1] \theta_1} \cdots e^{[S_n] \theta_n} M + \cdots + e^{[S_1] \theta_1} \cdots [S_n] \dot{\theta}_n e^{[S_n] \theta_n} M \]
Computing the Jacobian (1)

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• Take the derivative:

\[ \dot{T}(\theta) = \frac{d}{dt} \left( e^{[s_1]_1}e^{[s_2]_2} \cdots e^{[s_n]_n}M \right) + \cdots + e^{[s_1]_1}e^{[s_2]_2} \cdots \frac{d}{dt} \left( e^{[s_n]_n}M \right) \]

\[ = [s_1] \dot{\theta}_1 e^{[s_1]_1} \cdots e^{[s_n]_n}M + \cdots + e^{[s_1]_1} \cdots [s_n] \dot{\theta}_n e^{[s_n]_n}M \]

• Take the inverse:

\[ T^{-1}(\theta) = M^{-1} e^{-[s_n]_n} \cdots e^{-[s_1]_1} \]
Computing the Jacobian (1)

- Recall forward kinematics:
  \[ T(\theta) = e^{[s_1]_1} e^{[s_2]_2} \ldots e^{[s_n]_n} M \]

- Take the derivative:
  \[ \dot{T}(\theta) = \frac{d}{dt} \left( e^{[s_1]_1} e^{[s_2]_2} \ldots e^{[s_n]_n} M + \ldots + e^{[s_1]_1} e^{[s_2]_2} \frac{d}{dt} \left( e^{[s_n]_n} \right) M \right) \]
  \[ = [s_1] \dot{\theta}_1 e^{[s_1]_1} \ldots e^{[s_n]_n} M + \ldots + e^{[s_1]_1} [s_n] \dot{\theta}_n e^{[s_n]_n} M \]

- Take the inverse:
  \[ T^{-1}(\theta) = M^{-1} e^{-[s_n]_n} \ldots e^{-[s_1]_1} \]

- Recall: \[ [V_s] = \dot{T} T^{-1} \]
  \[ \dot{T} T^{-1} = [s_1] \dot{\theta}_1 + e^{[s_1]_1} [s_2] e^{-[s_1]_1} \dot{\theta}_2 + e^{[s_1]_1} e^{[s_2]_2} [s_3] e^{-[s_2]_2} e^{-[s_1]_1} \dot{\theta}_3 + \ldots \]
Computing the Jacobian (2)

• Expressed via Adjoint:

\[ \mathcal{V}_s = S_1 \dot{\theta}_1 + Ad_{e[S_1]\theta_1}(S_2) \dot{\theta}_2 + Ad_{e[S_1]\theta_1 e[S_2]\theta_2}(S_3) \dot{\theta}_3 + \ldots \]

• Gives our Space Jacobian:

\[ \mathcal{V}_s = J_s(\theta) \dot{\theta} \]

\[ J_s(\theta) = [J_{s1}, J_{s2}, \ldots, J_{sn}] \]

\[ J_{s1} = S_1, \quad J_{si} = Ad_{e[S_i] \theta} \ldots e[S_{i-1}] \theta_{i-1}(S_i) \]

• **Intuition:** If \( T_i M \) is the configuration when you set joints \( \theta_1, \ldots, \theta_i \) and leave remaining joints at 0, then \( J_{si} \) is the screw vector at joint \( i \), for an arbitrary \( \theta \).
Visualizing the Jacobian

- By inspection, let’s find $J_{s3}$
- Ignore joints $\theta_3, \theta_4, \theta_5$, as they do not displace axis 3 relative to $\{s\}$
- If $\theta_1 = 0$ and $\theta_2$ is arbitrary, then $T_{ss'} = e^{[S_2]\theta_2}$
- If $\theta_1$ is arbitrary, then $T_{ss''} = e^{[S_1]\theta_1}e^{[S_2]\theta_2}$
- $J_{s3} = [Ad_{T_{ss''}}]S_3 = [Ad_{e^{[S_1]\theta_1}e^{[S_2]\theta_2}}]S_3$
Summary

• Defined **velocity kinematics** as means to compute the twist at the end-effector frame

• Interpreted the end-effector twist as a way to map actuator limits to **possible end-effector velocities** (directions of motion) and insight as to how close a pose is to a **singularity**

• **Next time:**
  • Return to FK for URDF tutorial
  • Then on to VK examples