lecture 9
quick recap and
velocity kinematics I

Prof. DC

February 14, 2023 <3

Modern Robotics Ch. 5.1-5.3

Admin

- HW4 due date moved to after exam 1
- Exam 1 next week (Mon – Wed)
  - **Remember to signup for the exam through the CBTF website**
  - Practice exam available on prairielearn
  - I recommend going through past homework
  - Review lecture on Thursday
  - No lecture on Tuesday 2/21

Who is Carl Gustav Jacob Jacobi?

- A German mathematician who contributed to elliptic functions, dynamics, differential equations, determinants, and number theory
- In Germany during the Revolution of 1848, Jacobi was politically involved with the Liberal club, which after the revolution ended led to his royal grant being temporarily cut off
- His PhD student, Otto Hesse, is known for the Hessian Matrix (second-order partial derivatives of a scalar-valued function, or scalar field)

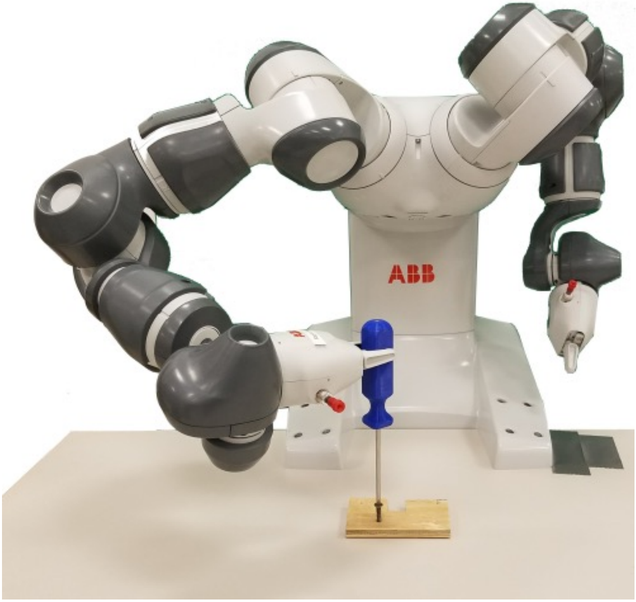
Credit: Wikipedia

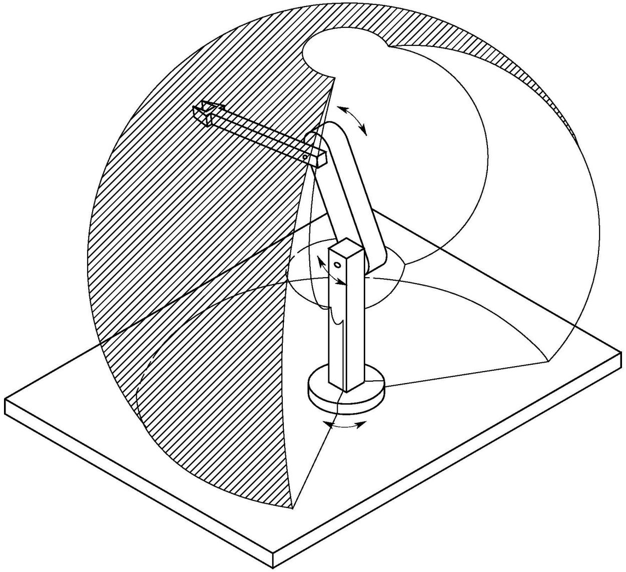
Quick Course Recap

- We use our knowledge of the links, joints, and actuators to represent the **configuration** of the robot
- The number of **degrees of freedom** of the robot is the smallest number of coordinates needed to represent the configuration

Quick Course Recap

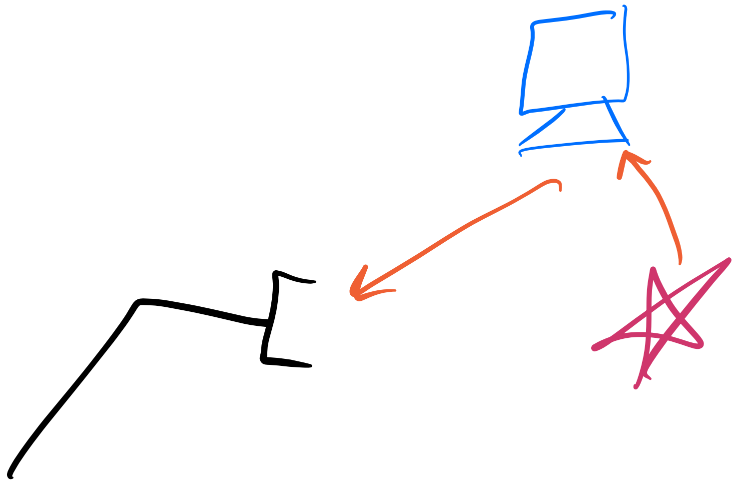
- We use our knowledge of the links, joints, and actuators to represent the **configuration** of the robot
- The number of **degrees of freedom** of the robot is the smallest number of coordinates needed to represent the configuration
- The **task space** is the space in which the robot's task can be naturally expressed
- The **workspace** is the specification of the configurations that the end-effector of the robot can reach





Rotations and Transformations

- What do these matrices represent and what operations can they do?



Rotations and Transformations

- What do these matrices represent and what operations can they do?

- Exponential coordinates allow us to parameterize rotations by the rotation axis and angle of rotation and homogeneous transformations by the screw axis

Velocity Kinematics

Forward Kinematics

Calculate the position of the end-effector of an open chain given joint angles

$$\underbrace{x(t)}_{\text{pose of tool}} = f(\underbrace{\theta(t)}_{\text{joint angles}})$$

→ forward kinematics $T(\theta)$

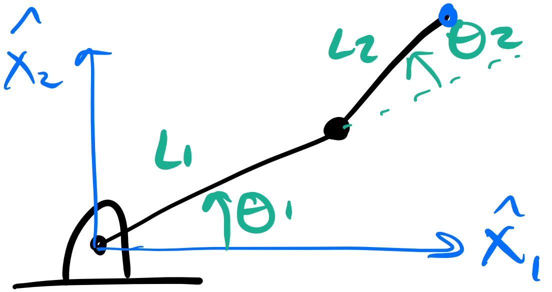
Velocity Kinematics

Calculate the velocity (twist!) of the end-effector frame

$$\dot{x} = \frac{d}{dt} x(t) = \underbrace{\frac{\partial f}{\partial \theta}(\theta)}_{J(\theta)} \dot{\theta}$$

$J(\theta) \rightarrow$ jacobian of f

Example



FK: $x_1 = L_1\cos\theta_1 + L_2\cos(\theta_1+\theta_2)$

$x_2 = L_1\sin\theta_1 + L_2\sin(\theta_1+\theta_2)$

Differentiate:

$\dot{x}_1 = -L_1\dot{\theta}_1\sin\theta_1 - L_2(\dot{\theta}_1+\dot{\theta}_2)\sin(\theta_1+\theta_2)$

$\dot{x}_2 = L_1\dot{\theta}_1\cos\theta_1 + L_2(\dot{\theta}_1+\dot{\theta}_2)\cos(\theta_1+\theta_2)$

rearrange:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \overbrace{\left[\underbrace{\begin{matrix}-L_1\sin\theta_1 - L_2\sin(\theta_1+\theta_2)\\ L_1\cos\theta_1 + L_2\cos(\theta_1+\theta_2)\end{matrix}}_{J_1(\theta)} \quad \underbrace{\begin{matrix}-L_2\sin(\theta_1+\theta_2)\\ L_2\cos(\theta_1+\theta_2)\end{matrix}}_{J_2(\theta)}\right]}^{J(\theta)} \begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\end{bmatrix}$$

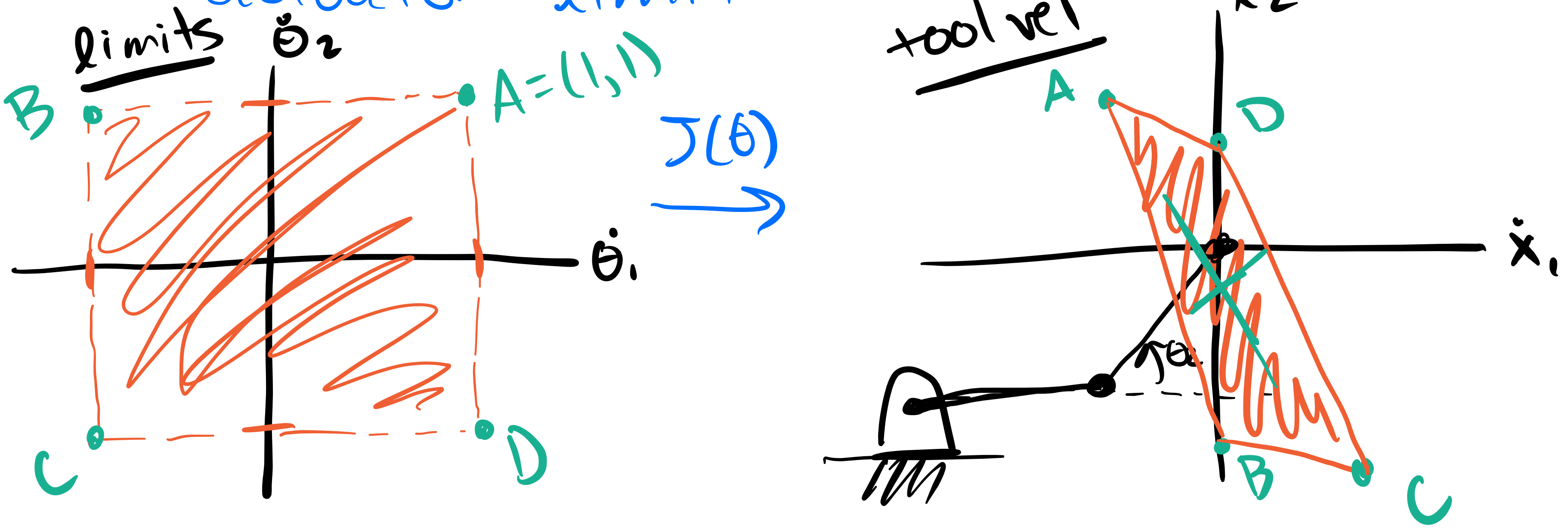
$$\dot{x} = J_1(\theta)\dot{\theta}_1 + J_2(\theta)\dot{\theta}_2$$

Jacobian Mapping

if $L_1 = L_2 = 1$, $\theta_1 = 0$, $\theta_2 = \pi/4$, then:

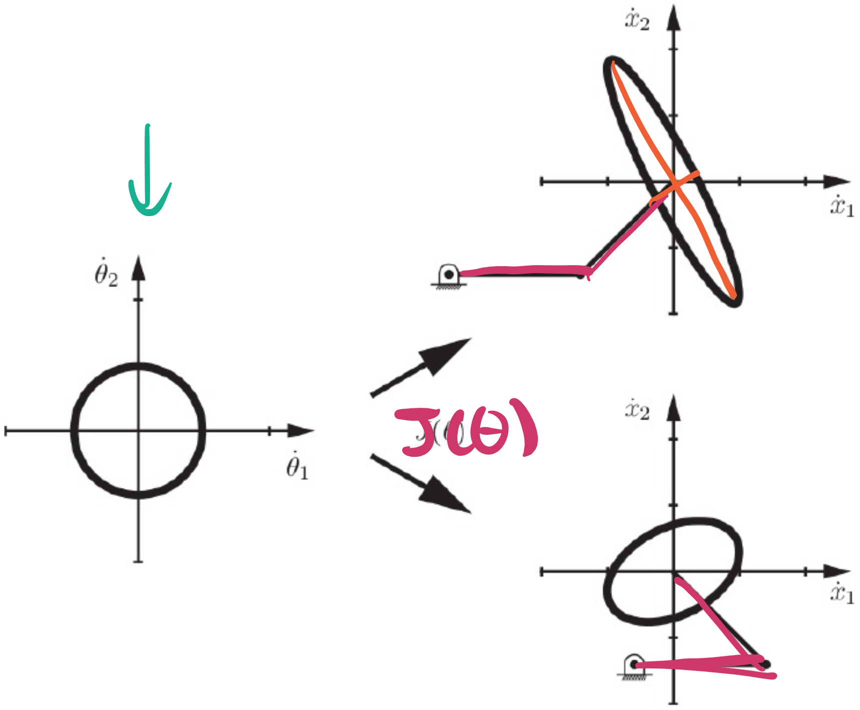
$$J\left(\begin{bmatrix} 0 \\ \pi/4 \end{bmatrix}\right) = \begin{bmatrix} -.71 & -.71 \\ 1.71 & .71 \end{bmatrix}$$

assume actuator limits: $\dot{\theta}_i \in [-1, 1]$



Jacobian and Manipulability

- Suppose joint limits are $\dot{\theta}_1^2 + \dot{\theta}_2^2 \leq 1$
- The ellipsoid obtained by mapping through the Jacobian is called **the manipulability ellipsoid**



Jacobian and Statics

- Assume a force f_{tip} is applied on the end-effector. What torque (τ) must be applied at the joint to keep the fixed position?

Jacobian and Statics

- Assume a force f_{tip} is applied on the end-effector. What torque (τ) must be applied at the joint to keep the fixed position?
- By conservation of power:
$$\underline{f_{tip}^{\top} v_{tip}} = \underline{\tau^{\top}}\dot{\theta}, \qquad \forall \dot{\theta}$$
- Since $v_{tip} = J(\theta)\dot{\theta}$, we have
$$f_{tip}^{\top} \underline{J(\theta)\dot{\theta}} = \tau^{\top}\dot{\theta}, \qquad \forall \dot{\theta}$$
- Which gives:
$$\tau = J^{\top}(\theta) f_{tip}$$

Jacobian and Statics

- Assume a force f_{tip} is applied on the end-effector. What torque (τ) must be applied at the joint to keep the fixed position?
- By conservation of power:

$$f_{tip}^{\top} v_{tip} = \tau^{\top}\dot{\theta}, \qquad \forall\dot{\theta}$$

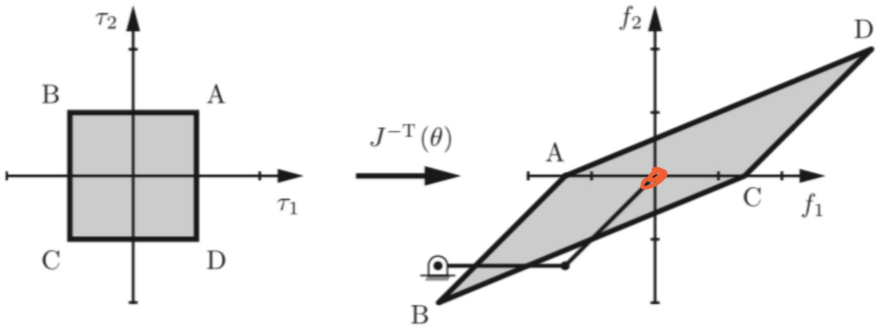
- Since $v_{tip} = J(\theta)\dot{\theta}$, we have

$$f_{tip}^{\top} J(\theta)\dot{\theta} = \tau^{\top}\dot{\theta}, \qquad \forall\dot{\theta}$$

- Which gives:

$$\tau = J^{\top}(\theta) f_{tip}$$

- If $J^{\top}(\theta)$ is invertible, then given the limits of the torques at the joints, we can compute all the forces that can be counteracted at the end-effector



Computing the Jacobian (1)

- Recall forward kinematics:

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} \dots e^{[\mathcal{S}_n]\theta_n} M$$

Computing the Jacobian (1)

- Recall forward kinematics:

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} \dots e^{[\mathcal{S}_n]\theta_n} M$$

- Take the derivative:

$$\begin{aligned}\dot{T}(\theta) &= \frac{d}{dt}\left(e^{[\mathcal{S}_1]\theta_1}\right) e^{[\mathcal{S}_2]\theta_2} \dots e^{[\mathcal{S}_n]\theta_n} M + \dots + e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} \dots \frac{d}{dt}\left(e^{[\mathcal{S}_n]\theta_n}\right) M \\ &= [\mathcal{S}_1]\,\dot{\theta}_1 e^{[\mathcal{S}_1]\theta_1} \dots e^{[\mathcal{S}_n]\theta_n} M + \dots + e^{[\mathcal{S}_1]\theta_1} \dots [\mathcal{S}_n]\,\dot{\theta}_n e^{[\mathcal{S}_n]\theta_n} M\end{aligned}$$

Computing the Jacobian (1)

- Recall forward kinematics:

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} \dots e^{[\mathcal{S}_n]\theta_n} M$$

- Take the derivative:

$$\begin{aligned}\dot{T}(\theta) &= \frac{d}{dt}\left(e^{[\mathcal{S}_1]\theta_1}\right) e^{[\mathcal{S}_2]\theta_2} \dots e^{[\mathcal{S}_n]\theta_n} M + \dots + e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} \dots \frac{d}{dt}\left(e^{[\mathcal{S}_n]\theta_n}\right) M \\ &= [\mathcal{S}_1]\,\dot{\theta}_1 e^{[\mathcal{S}_1]\theta_1} \dots e^{[\mathcal{S}_n]\theta_n} M + \dots + e^{[\mathcal{S}_1]\theta_1} \dots [\mathcal{S}_n]\,\dot{\theta}_n e^{[\mathcal{S}_n]\theta_n} M\end{aligned}$$

- Take the inverse:

$$T^{-1}(\theta) = M^{-1} e^{-[\mathcal{S}_n]\theta_n} \dots e^{-[\mathcal{S}_1]\theta_1}$$

Computing the Jacobian (1)

- Recall forward kinematics:

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} \dots e^{[\mathcal{S}_n]\theta_n} M$$

- Take the derivative:

$$\dot{T}(\theta) = \frac{d}{dt}\left(e^{[\mathcal{S}_1]\theta_1}\right) e^{[\mathcal{S}_2]\theta_2} \dots e^{[\mathcal{S}_n]\theta_n} M + \dots + e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} \dots \frac{d}{dt}\left(e^{[\mathcal{S}_n]\theta_n}\right) M$$

$$= [\mathcal{S}_1]\,\dot{\theta}_1 e^{[\mathcal{S}_1]\theta_1} \dots e^{[\mathcal{S}_n]\theta_n} M + \dots + e^{[\mathcal{S}_1]\theta_1} \dots [\mathcal{S}_n]\,\dot{\theta}_n e^{[\mathcal{S}_n]\theta_n} M$$

- Take the inverse:

$$T^{-1}(\theta) = M^{-1} e^{-[\mathcal{S}_n]\theta_n} \dots e^{-[\mathcal{S}_1]\theta_1}$$

- Recall: $[\mathcal{V}_s] = \dot{T}\, T^{-1}$

$$\dot{T}T^{-1} = [\mathcal{S}_1]\dot{\theta}_1 + e^{[\mathcal{S}_1]\theta_1}[\mathcal{S}_2]e^{-[\mathcal{S}_1]\theta_1}\dot{\theta}_2 + e^{[\mathcal{S}_1]\theta_1}e^{[\mathcal{S}_2]\theta_2}[\mathcal{S}_3]e^{-[\mathcal{S}_2]\theta_2}e^{-[\mathcal{S}_1]\theta_1}\dot{\theta}_3 + \cdots$$

Computing the Jacobian (2)

- Expressed via Adjoint:

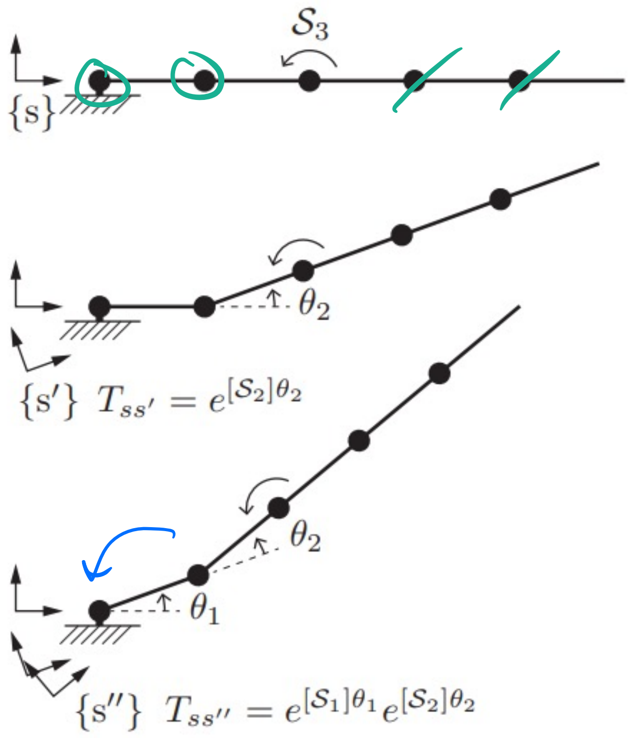
$$\mathcal{V}_s = \underbrace{\mathcal{S}_1\dot{\theta}_1}_{J_{s1}} + \underbrace{Ad_{e^{[\mathcal{S}_1]\theta_1}}(\mathcal{S}_2)}_{J_{s2}}\dot{\theta}_2 + \underbrace{Ad_{e^{[\mathcal{S}_1]\theta_1}e^{[\mathcal{S}_2]\theta_2}}(\mathcal{S}_3)}_{J_{s3}}\dot{\theta}_3 + \cdots$$

- Gives our Space Jacobian: $\mathcal{V}_s = J_s(\theta)\dot{\theta}$, $J_s(\theta) = [\, J_{s1} \quad J_{s2} \cdots J_{sn} \,]$

$J_{s1} = \mathcal{S}_1$, $J_{si} = Ad_{e^{[\mathcal{S}_1]\theta} \cdots e^{[\mathcal{S}_{i-1}]\theta_{i-1}}}(\mathcal{S}_i)$

- **Intuition:** If $T_i M$ is the configuration when you set joints $\theta_1, \ldots, \theta_i$ and leave remaining joints at 0, then J_{si} is the screw vector at joint i, for an arbitrary θ

Visualizing the Jacobian



- By inspection, let's find J_{s3}
- Ignore joints $\theta_3, \theta_4, \theta_5$, as they do not displace axis 3 relative to {s}
- If $\theta_1 = 0$ and θ_2 is arbitrary, then $T_{ss'} = e^{[\mathcal{S}_2]\theta_2}$
- If θ_1 is arbitrary, then $T_{ss''} = e^{[\mathcal{S}_1]\theta_1}e^{[\mathcal{S}_2]\theta_2}$
- $J_{s3} = [Ad_{T_{ss''}}]\mathcal{S}_3 = [Ad_{e^{[\mathcal{S}_1]\theta_1}e^{[\mathcal{S}_2]\theta_2}}]\mathcal{S}_3$

Summary

- Defined **velocity kinematics** as means to compute the twist at the end-effector frame
- Interpreted the end-effector twist as a way to map actuator limits to **possible end-effector velocities** (directions of motion) and insight as to how close a pose is to a **singularity**
- **Next time:**
  - Return to FK for URDF tutorial
  - Then on to VK examples