Lecture 8: Forward Kinematics II

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Modern Robotics Ch. 4
Admin

• Homework 3 FK question is now bonus
• Homework party today will be in **1232 CSL Studio**
• Some exam logistics:
  • CBTF signup open today (2/9)
  • Thursday lecture before exams (2/16) will be a TA led review session
  • No class or prof OH during the exam windows (2/21)
  • No HW due on exam weeks
  • Practice Exam will be posted next week
Denavit-Hartenberg Parameters

• In the 1950s, when Dick Hartenberg, a professor, and Jacques Denavit, a PhD student, developed a way to represent mathematically how mechanisms move

• They showed that the position of one link connected to another could be represented **minimally** using only four parameters
  • Known as the Denavit-Hartenberg (DH) parameters

• In 1981, Richard Paul demonstrated its value for the kinematic analysis of robotic systems

https://robotics.northwestern.edu/history.html
Screw Motions as Matrix Exponential

The screw axis $S_i \in \mathbb{R}^e$ can be expressed in matrix form as:

$$[S_i] = \begin{bmatrix} [\omega_i] & v \\ 0 & 0 \end{bmatrix} \in se(3)$$

To express a screw motion given a screw axis, we use the matrix exponential:

$$e^{[S]\theta} \in SE(3)$$

**Proposition 3.25.** Let $S = (\omega, v)$ be a screw axis. If $\|\omega\| = 1$ then, for any distance $\theta \in \mathbb{R}$ traveled along the axis,

$$e^{[S]\theta} = \begin{bmatrix} e^{[\omega]\theta} & (I\theta + (1 - \cos \theta)[\omega] + (\theta - \sin \theta)[\omega]^2) v \\ 0 & 1 \end{bmatrix}. \quad (3.88)$$

If $\omega = 0$ and $\|v\| = 1$, then

$$e^{[S]\theta} = \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix}.$$
Case 1: Revolute Joint

- $\|\omega\| = 1$
- $v = -\omega \times q + h\omega$
- $v = -\omega \times q$
Case 2: Prismatic Joint

- $\|\omega\| = 0$
- $\|v\| = 1$
- Axis of movement defines $v$
Product of Exponentials Approach

Let each joint $i$ have a configuration defined by $\theta_i$

Initialization steps:

- Choose a fixed frame $\{s\}$
- Choose an end-effector (tool) frame attached to the robot $\{b\}$
- Put all joints in zero position
- Let $M \in SE(3)$ be the configuration of $\{b\}$ in the $\{s\}$ frame when the robot is in the zero position
Product of Exponentials Approach

• Given zero position $M$
• For each joint $i$, define the screw axis
• For each motion of a joint, define the screw motion
• These operations compose nicely through multiplication, giving us the Product of Exponentials (PoE) formula!

$$T(\theta) = e^{[S_1] \theta_1} e^{[S_2] \theta_2} \ldots e^{[S_{n-1}] \theta_{n-1}} e^{[S_n] \theta_n} M$$
Example 1

1. Find $M$

$$M = \begin{bmatrix} R_0 & P_0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & L_1 + L_2 + L_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2. Find $w_i \rightarrow$ all joints are revolute!

$$w_i = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \text{ A joints}$$

3. Compute $V_i$

$$V_3 = -w_3 \times q_3, q_3 = [L_1 + L_2, 0, 0]^T$$

$$= \begin{bmatrix} 0 \\ -(L_1 + L_2) \\ 0 \end{bmatrix}$$

$$S_3 = \begin{bmatrix} w_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -(L_1 + L_2) \end{bmatrix}$$

4. $axb = \begin{bmatrix} a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \end{bmatrix}$

*Figure 4.1:* Forward kinematics of a 3R planar open chain. For each frame, the $x$- and $y$-axis is shown; the $z$-axes are parallel and out of the page.
Example 1: PoE

Compute $e^{[s_i] \theta_i}$ for each joint:

$$e^{[s_i] \theta_i} = \begin{bmatrix} e^{[\omega_i] \theta_i} & (I \theta_i + (1 - \cos \theta_i)[\omega_i] + (\theta_i - \sin \theta_i)[\omega_i]^2) v_i \\ 0 & 1 \end{bmatrix}$$

and compose with $M$:

$$T(\theta) = e^{[s_1] \theta_1} e^{[s_2] \theta_2} e^{[s_3] \theta_3} M$$
Example 2

\[
M = \begin{bmatrix}
0 & 0 & 0 & L_1 \
0 & 0 & 0 & -L_2 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

1. \( \omega_1 = [0, 0, 1] \), \( \mathbf{v}_1 = [0] \)
2. \( \omega_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \), \( \mathbf{v}_2 = -\omega_2 \times \mathbf{q}_2 \)
\( \mathbf{q}_2 = [L_1, 0, 0]^T \)
\( \mathbf{v}_2 = [0, 0, 0, -L_1]^T \)
3. \( \omega_3 = [0, 0, 1] \), \( \mathbf{v}_3 = -\omega_3 \times \mathbf{q}_3 \)
\( \mathbf{q}_3 = [L_1, 0, -L_2]^T \)
\( \mathbf{v}_3 = [0, -L_2, 0]^T \)

Figure 4.3: A 3R spatial open chain.
Example 3

\[ M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

1. \( \omega_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \), \( \nu_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \)

2. \( q_1 = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} \), \( v_1 = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} \)

3. \( \omega_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \), \( v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \)

4. \( \omega_3 = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \), \( q_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \)

5. \( v_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \)

6. \( S = [ S_1, S_2, S_3 ] \)
So Far: Spatial Forward Kinematics

- Initialization steps:
  - Choose a fixed frame \(\{s\}\)
  - Choose an end-effector (tool) frame \(\{b\}\)
  - Put all joints in zero position
  - Let \(M \in SE(3)\) be the configuration of \(\{b\}\) in the \(\{s\}\) frame when the robot is in the zero position

- For each joint \(i\), define the screw axis
- For each motion of a joint, define the screw motion
- These operations compose nicely through multiplication, giving us the Product of Exponentials (PoE) formula!

Figure 4.2: Illustration of the PoE formula for an \(n\)-link spatial open chain.
PoE in the End-Effector Frame

 recalled: $e^{M^{-1}PM} = M^{-1}e^PM \rightarrow e^PM = Me^{M^{-1}PM}$

$T(\theta) = e^{[S_i]\theta_i} \cdots e^{[S_n]\theta_n}M$

$= e^{[S_i]\theta_i} \cdots Me^{M^{-1}[S_n]JM\theta_n}$

$= e^{[S_i]\theta_i} \cdots Me^{M^{-1}[S_n]JM\theta_n}$

$= Me^{M^{-1}[S_i]JM\theta_i} \cdots e^{M^{-1}[S_n]JM\theta_n}$

What is $M^{-1}[S_i]JM$? $\rightarrow [B_i]$ screw axis in body frame

Body Form of PoE: $T(\theta) = Me^{[B_i]\theta_i} \cdots e^{[B_n]\theta_n}$
Body Form Example

\[ M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

1. \( \omega_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix} \), \( \theta_1 = \begin{bmatrix} -2 \\ -2 \\ 0 \end{bmatrix} \)

\( v_1 = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} \)

2. \( \| \omega_2 \| = 0 \), \( v_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \)

3. \( \| \omega_3 \| = 0 \), \( v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \)

4. \( \omega_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \), \( \theta_4 = \begin{bmatrix} -2 \\ 0 \end{bmatrix} \)

\( v_4 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \)
Summary

• Learned the basics of **forward kinematics**, which gives us a model for computing position and orientation of the end-effector.

• Uses the **product of exponentials formula** to define this transformation as composed matrix multiplications.

• Our **fancy screw motion matrix exponential** is just another way to write down **homogenous transformations in three dimensions**!

• Our usual homogenous transformation matrices can also be used for forward kinematics (the Denavit-Hartenberg (DH) representation):
  • We define a frame for each link in the frame of the previous link. So to compute the position of the end effector, a frame for each link must be defined in terms of the previous link.

• **With screw motions, we have only two reference frames** (base and tool), and then each joint screw motion is defined in the base frame.