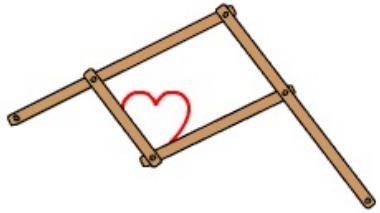
Lecture 8: Forward Kinematics II

Prof. DC

Feb. 9, 2023

Modern Robotics Ch. 4



Admin

- Homework 3 FK question is now bonus
- Homework party today will be in **1232 CSL Studio**
- Some exam logistics:
  - CBTF signup open today (2/9)
  - Thursday lecture before exams (2/16) will be a TA led review session
  - No class or prof OH during the exam windows (2/21)
  - No HW due on exam weeks
  - Practice Exam will be posted next week

Denavit-Hartenberg Parameters

- In the 1950s, when Dick Hartenberg, a professor, and Jacques Denavit, a PhD student, developed a way to represent mathematically how mechanisms move
- They showed that the position of one link connected to another could be represented **minimally** using only four parameters
  - Known as the Denavit-Hartenberg (DH) parameters
- In 1981, Richard Paul demonstrated its value for the kinematic analysis of robotic systems

https://robotics.northwestern.edu/history.html

Screw Motions as Matrix Exponential

The screw axis $\mathcal{S}_i$ $\in \mathbb{R}^6$ can be expressed in matrix form as:

$$[\mathcal{S}_i] = \begin{bmatrix} [\omega_i] & v \\ 0 & 0 \end{bmatrix} \in se(3)$$

To express a **screw motion** given a screw axis, we use the matrix exponential:

$$e^{[\mathcal{S}]\theta} \in SE(3)$$

Proposition 3.25. *Let $\mathcal{S} = (\omega, v)$ be a screw axis. If $\|\omega\| = 1$ then, for any distance $\theta \in \mathbb{R}$ traveled along the axis,*

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$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} e^{[\omega]\theta} & \left(I\theta + (1-\cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2\right) v \\ 0 & 1 \end{bmatrix}. \tag{3.88}$$

If $\omega = 0$ and $\|v\| = 1$, then

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix}.$$

Modeling Robot Joints as Screw Motions

Case 1: Revolute Joint

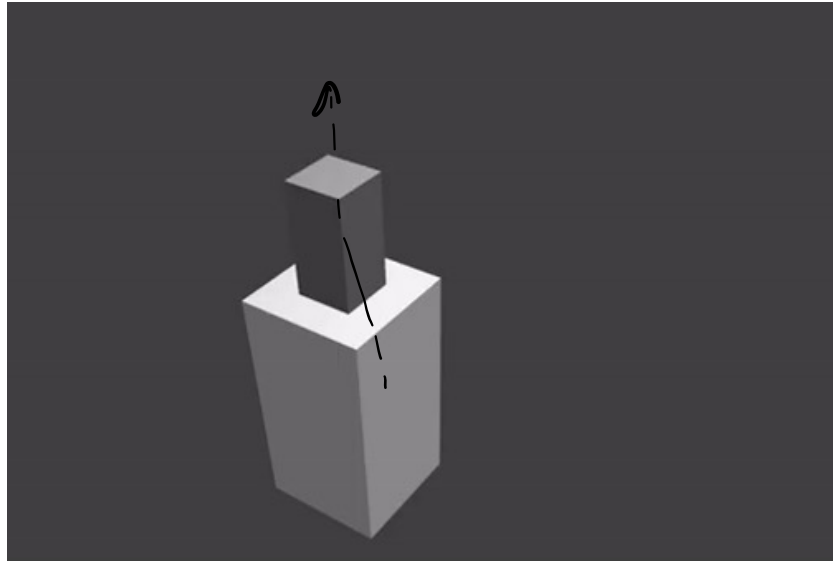
- $\|\omega\| = 1$
- $v = -\omega \times q + h\omega$
- $v = -\omega \times q$

$v = \cancel{h\omega} - \omega \times q$

Modeling Robot Joints as Screw Motions

Case 2: Prismatic Joint

- $\|\omega\| = 0$
- $\|v\| = 1$
- Axis of movement defines v



Product of Exponentials Approach

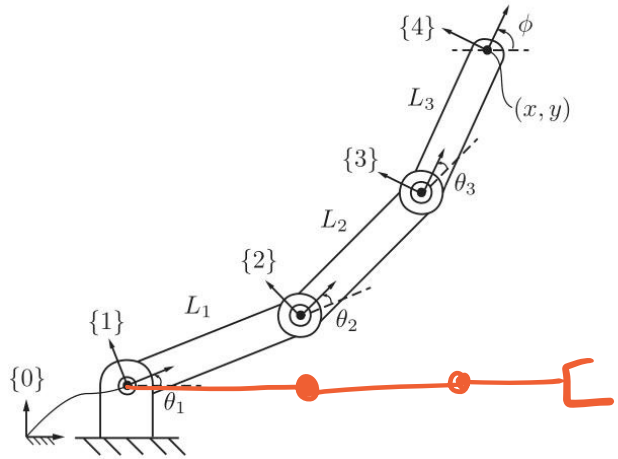


Figure 4.1: Forward kinematics of a 3R planar open chain. For each frame, the $\hat{x}$- and $\hat{y}$-axis is shown; the $\hat{z}$-axes are parallel and out of the page.

Let each joint i have a configuration defined by θ_i

Initialization steps:

- ✓ Choose a fixed frame $\{s\}$
- ✓ Choose an end-effector (tool) frame attached to the robot $\{b\}$
- ✓ Put all joints in *zero position* M
- ✓ Let $M \in SE(3)$ be the configuration of $\{b\}$ in the $\{s\}$ frame when the robot is in the zero position

Product of Exponentials Approach

- Given zero position M
- For each joint i, define the screw axis
- For each motion of a joint, define the screw motion
- These operations compose nicely through multiplication, giving us the Product of Exponentials (PoE) formula!

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} \dots e^{[\mathcal{S}_{n-1}]\theta_{n-1}} e^{[\mathcal{S}_n]\theta_n} M$$

Example 1

① Find M

$$M = \begin{bmatrix} R_0 & p_0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & L_1+L_2+L_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

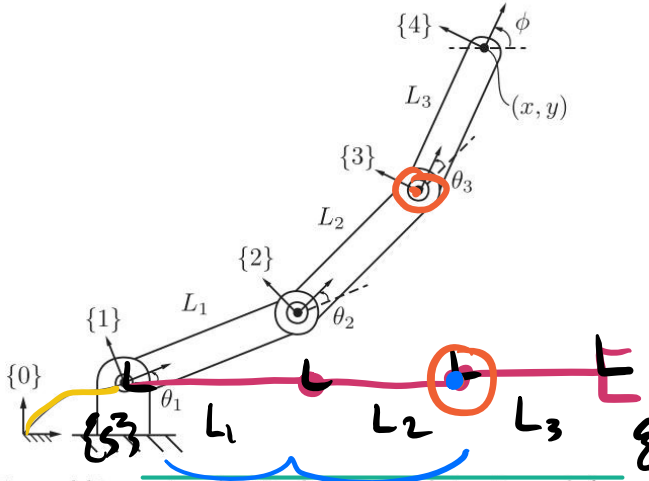


Figure 4.1: Forward kinematics of a 3R planar open chain. For each frame, the $\hat{x}$- and $\hat{y}$-axis is shown; the $\hat{z}$-axes are parallel and out of the page.

② find ω_i → all joints are revolute!

$$\omega_i = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \forall \text{ joints } i$$

③ compute v_i

$$v_3 = -\omega_3 \times q_3, \quad q_3 = [L_1+L_2, 0, 0]^T$$

$$= \begin{bmatrix} 0 \\ -(L_1+L_2) \\ 0 \end{bmatrix}$$

$$S_3 = \begin{bmatrix} \omega_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -(L_1+L_2) \\ 0 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 0 \\ -L_1 \\ 0 \end{bmatrix}$$

$$v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$a \times b = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

Example 1: PoE

Compute $e^{[\mathcal{S}_i]\theta_i}$ for each joint:

$$e^{[\mathcal{S}_i]\theta_i} = \begin{bmatrix} e^{[\omega_i]\theta_i} & (I\theta_i + (1-\cos\theta_i)[\omega_i] + (\theta_i - \sin\theta_i)[\omega_i]^2)v_i \\ 0 & 1 \end{bmatrix}$$

and compose with M:

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M$$

Example 2

$$M = \begin{array}{c} \begin{array}{ccc} \hat{x} & \hat{y} & \hat{z} \end{array} \\ \left[\begin{array}{ccc|c} 0 & 0 & 1 & L_1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -L_2 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

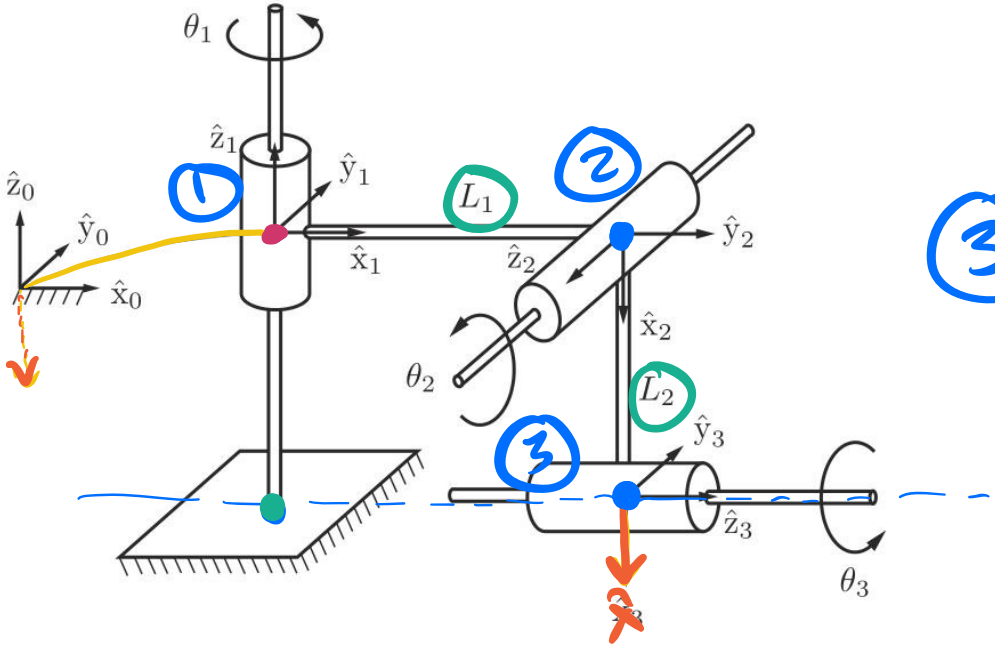


Figure 4.3: A 3R spatial open chain.

① $\omega_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

② $\omega_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$, $v_2 = -\omega_2 \times q_2$

$q_2 = [L_1, 0, 0]^T$

$v_2 = [0, 0, -L_1]^T$

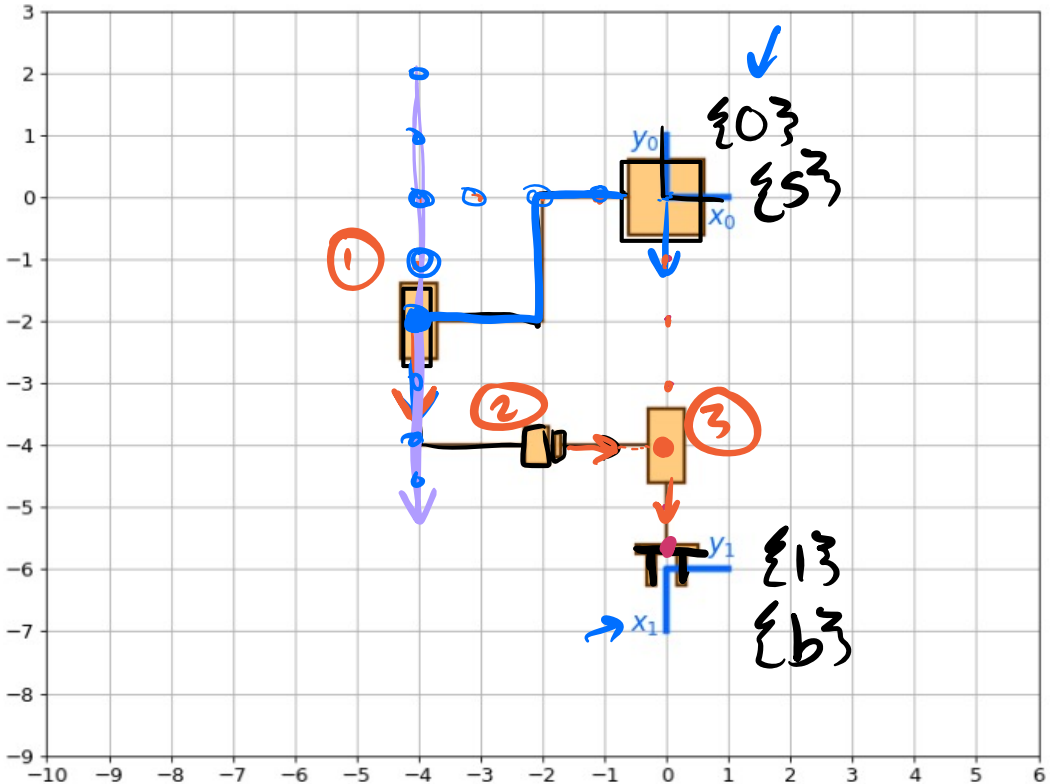
③ $\omega_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $v_3 = -\omega_3 \times q_3$

$q_3 = [\cancel{L_1}\,0, 0, -L_2]$

$v_3 = \begin{bmatrix} 0 \\ -L_2 \\ 0 \end{bmatrix}$

Example 3

$$M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



① $\omega_1 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$, $v_1 = -\omega_1 \times q_1$

$q_1 = [-4, -2, 0]^T$

$v_1 = [0, 0, 4]^T$ ✓

② $\omega_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

③ $\omega_3 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$, $q_3 = \begin{bmatrix} 0 \\ -4 \\ 0 \end{bmatrix}$

$v_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$S = [\underbrace{S_1}_{\in \mathbb{R}^6} \quad S_2 \quad S_3]^{\in \mathbb{R}^{6\times 3}}$$

So Far: Spatial Forward Kinematics

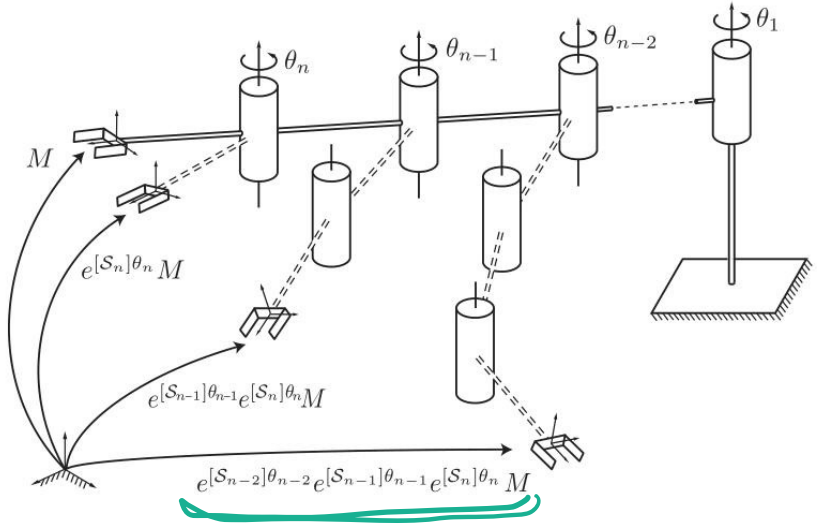


Figure 4.2: Illustration of the PoE formula for an n-link spatial open chain.

- Initialization steps:
  - Choose a fixed frame $\{s\}$
  - Choose an end-effector (tool) frame $\{b\}$
  - Put all joints in *zero position*
  - Let $M \in SE(3)$ be the configuration of $\{b\}$ in the $\{s\}$ frame when the robot is in the zero position
- For each joint i, define the screw axis
- For each motion of a joint, define the screw motion
- These operations compose nicely through multiplication, giving us the Product of Exponentials (PoE) formula!

PoE in the End-Effector Frame

recall: $e^{M^{-1}PM} = M^{-1} e^{P} M \rightsquigarrow e^{P} M = M e^{M^{-1}PM}$

$$T(\theta) = e^{[S_1]\theta_1} \cdots e^{[S_n]\theta_n} M$$

$$= e^{[S_1]\theta_1} \cdots M e^{M^{-1}[S_n]M\theta_n}$$

$$= e^{[S_1]\theta_1} \cdots M e^{M^{-1}[S_{n-1}]M\theta_{n-1}} e^{M^{-1}[S_n]M\theta_n}$$

$$= M e^{M^{-1}[S_1]M\theta_1} \cdots e^{M^{-1}[S_n]M\theta_n}$$

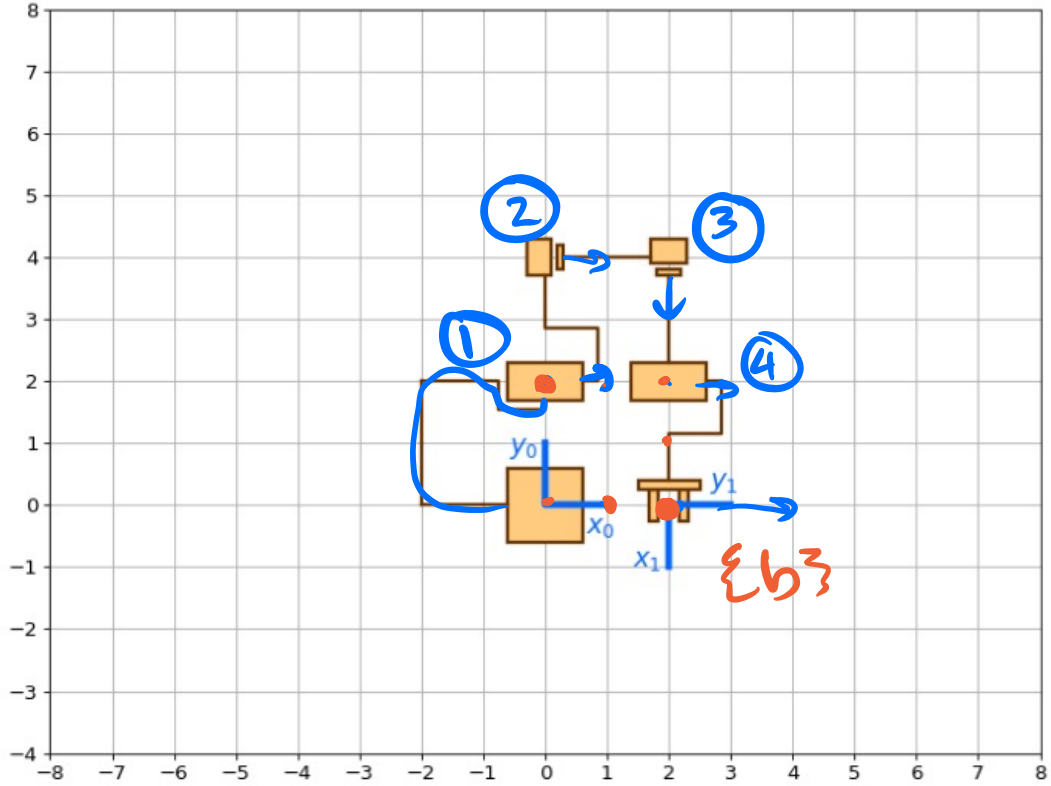
What is $M^{-1}[S_i]M$? $\rightsquigarrow [B_i]$ screw axis in body frame

Body Form of PoE:

$$T(\theta) = M e^{[B_1]\theta_1} \cdots e^{[B_n]\theta_n}$$

Body Form Example

$$M = \begin{bmatrix} 0 & 1 & 0 & 2 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



① $\omega_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $q_1 = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}$

$v_1 = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$

② $\|\omega_2\| = 0$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

③ $\|\omega_3\| = 0$, $v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

④ $\omega_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $q_4 = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$

$v_4 = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$

Summary

- Learned the basics of **forward kinematics**, which gives us a model for computing position and orientation of the end-effector
- Uses the **product of exponentials formula** to define this transformation as composed matrix multiplications
- Our **fancy screw motion matrix exponential** is just another way to write down **homogenous transformations in three dimensions!**
- Our usual homogenous transformation matrices can also be used for forward kinematics (the Denavit-Hartenberg (DH) representation)
  - We define a frame for each link in the frame of the previous link. So to compute the position of the end effector, a frame for each link must be defined in terms of the previous link
- **With screw motions, we have only two reference frames** (base and tool), and then each joint screw motion is defined in the base frame