Lecture 06: twists, screws, and wrenches

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Feb 2, 2023
Who is Michael Chasles?

• Rodrigues and Chasles took the entrance exam to Polytechnique/Normale at the same time, finishing first and second respectively
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• The proof that a spatial displacement can be decomposed into a rotation and slide around and along a line is attributed to the astronomer and mathematician Giulio Mozzi (1763),
  • In Italy, the screw axis is traditionally called asse di Mozzi

• However, most textbooks refer to a subsequent similar work by Michel Chasles dating 1830

• Several other scholars/contemporaries of M. Chasles obtained the same or similar results around that time, including G. Giorgini, Cauchy, Poinsot, Poisson, and Rodrigues
Mobile arm example

• Robot arm mounted on wheeled platform. Camera fixed to ceiling.

• \{b\} is body frame, \{c\} end-effector frame, \{e\} frame of object, and \{a\} is fixed frame.

• We assume the camera position and orientation in \{a\} is given.

• From camera measurements, you can evaluate the position and orientation of the body and the object in the camera frame.

• Since we designed our robot and have joint-angle estimates, we can obtain the end-effector position and orientation in the body frame.

• To pick up the object, we need the object position and orientation in the frame of our end-effector.
What is \( T_{ce} \)?

\[
T_{ab} T_{bc} T_{ce} = T_{ad} T_{de}
\]

\[
T_{ad} T_{ab} T_{bc} T_{ce} = T_{ad} T_{de}
\]

\[
T_{ce} = (T_{ad} T_{ab} T_{bc})^{-1} T_{ad} T_{de}
\]
Moving Frames: linear and angular velocity (1)

\[
T_{sb}(t) = T(t) = \begin{bmatrix} R(t) & \rho(t) \\ 0 & 1 \end{bmatrix} \quad \Rightarrow \quad \dot{T}(t) = \begin{bmatrix} \dot{R} & \dot{\rho} \\ 0 & 0 \end{bmatrix}
\]

\[
T^{-1}\dot{T} = \begin{bmatrix} R^T & -R^T\rho \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{R} & \dot{\rho} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \dot{R}R^T & R^T\dot{\rho} \\ 0 & 0 \end{bmatrix}
\]

\[
\text{lin vel in } \mathbb{E}_3 = \begin{bmatrix} [\omega_b] & v_b \\ 0 & 0 \end{bmatrix}
\]

\[\omega_b \text{ and } v_b \text{ give us a 6-parameter rep of body twist} \rightarrow \text{ spatial velocities in } \mathbb{E}_3\]
Moving Frames: linear and angular velocity (2)

\[ V_b = \begin{bmatrix} 3 \times 1 \\ 3 \times 1 \\ 3 \times 1 \end{bmatrix} \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} \begin{bmatrix} 3 \times 3 \\ 3 \times 1 \\ 3 \times 1 \end{bmatrix} \epsilon \text{se}(3) \]

Bracket notation: \[ [V_b] = \begin{bmatrix} [\omega_b] \\ 0 \\ v_b \end{bmatrix} \begin{bmatrix} 3 \times 3 \\ 3 \times 1 \\ 3 \times 1 \end{bmatrix} \epsilon \text{se}(3) \]

Note: 1. Brackets generally mean matrix rep.
   \[ [Y_b] \] and \[ [\omega_b] \] are different ops.

2. \[ [Y_b] \] is not skew-symmetric
Linear and angular velocity in reference frame

\[ \dot{T}^{-1} = \begin{bmatrix} R & \dot{R} \dot{p} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^T & -R^T \dot{p} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R R^T & \dot{p} - R R^T \dot{p} \\ 0 & 0 \end{bmatrix} \]

linear vel in $\mathbb{R}^3$

spatial twist: spatial vel in $\mathbb{R}^3$

\[ \mathbf{v}_s = \begin{bmatrix} w_s \\ v_s \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} w_s \end{bmatrix} \\ \begin{bmatrix} v_s \end{bmatrix} \end{bmatrix} \]

\[ \mathbf{v}_s = \dot{p} - R R^T \dot{p} = \dot{p} - w_s \times \dot{p} \]
Physical interpretation of \( v_s \)

• Assume that a very large rigid body is attached to the frame \{b\}, and is large enough to contain the origin of \{s\}

• What is the velocity of the point of this body as the origin of \{s\}?

• There are two components:
  • \( \dot{\rho} \): the motion of the body
  • \( \omega_s \times (-p) \): the rotation of the body
Twists

\[
[v_b] = T^{-1} \hat{T}, \quad [v_s] = \hat{T}T^{-1}
\]

\[
[v_b] = T^{-1} [v_s]T, \quad [v_s] = T [v_b]T^{-1}
\]

\[
[v_s] = \begin{bmatrix}
R [w_b] R^T & -R [w_b] R^T p + R v_b \\
0 & 0 \\
o & 0
\end{bmatrix}
\]

A trick: \([w] p = w \times p = -p \times w = -[p] w\)

\[
[w_s] = \begin{bmatrix}
R & 0 \\
[p] R & R
\end{bmatrix} [w_b]
\]

→ allows us to change frame of ref for twists

→ adjoint map/representation
Adjoint Map associated with $T^{Ad_T(V)}$

$$\gamma' = \begin{bmatrix} R & 0 \\ \left[ \rho \right] & R & R \end{bmatrix} \gamma$$

By construction, $\gamma' = Ad_T(V) \iff [\gamma'] = T[\gamma]T^{-1}$

Properties:

1. $Ad_{T_1}(Ad_{T_2}(V)) = Ad_{T_1T_2}(V)$

2. $[Ad_{T^{-1}}] = [Ad_T]^{-1}$
The twist corresponding to the instantaneous motion of the chassis of a three-wheeled vehicle can be visualized as an angular velocity $w$ about the point $r$. 
Screw Motions
2D Screw Motions

• Any rigid motion in the plane can be represented by a rotation around a well-chosen center
• We can encode it with $(\beta, s_x, s_y)$, where $(s_x, s_y)$ is the position of the center of the rotation, and $\beta$ the angle
2D Screw Motions

• Any rigid motion in the plane can be represented by a rotation around a well-chosen center
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• Recall that the angular velocity \(\omega\) can be viewed as \(\hat{\omega}\dot{\theta}\), where \(\hat{\omega}\) is the unit rotation axis and \(\dot{\theta}\) is the rate of rotation
• A twist \(\mathcal{V}\) can be interpreted in terms of a screw axis \(S\) and a velocity \(\dot{\theta}\) about that axis
Screw motions in 3D

Chasles-Mozzi Theorem:
Any displacement in 3D can be represented by a rotation and translation about the same axis, referred to as a screw motion.

$q \in \mathbb{R}^3$ is any point along the axis

$\hat{s}$ is a unit vector in the direction of the axis

$h$ is the **screw pitch**, which is the ratio between the linear and angular speed along axis

We write this as $S = \{q, \hat{s}, h\}$
Screw motions and Twists

screw to twist:
\[ \mathbf{v} = \begin{bmatrix} \mathbf{w} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{s}} \mathbf{\Theta} \\ \mathbf{h} \hat{\mathbf{s}} \mathbf{\Theta} - \hat{\mathbf{s}} \mathbf{\Theta} \times \mathbf{q} \end{bmatrix} \]

- trans along screw axis
- linear motion at origin induced by rot of screw axis

twist to screw:
\[ \hat{s} = \sum_{\| \mathbf{w} \|} = \hat{\mathbf{w}}, \quad \mathbf{\Theta} = \| \mathbf{w} \|, \quad \mathbf{h} = \frac{\hat{\mathbf{w}}^T \mathbf{v}}{\hat{\mathbf{w}}^T \mathbf{v}} = \frac{\mathbf{w}^T \mathbf{v}}{\| \mathbf{w} \|^2} \]

if $\mathbf{w} \neq \mathbf{0}$, find $\mathbf{q}$:
\[ \mathbf{v} = \mathbf{h} \hat{s} \mathbf{\Theta} - \hat{s} \mathbf{\Theta} \times \mathbf{q} \]
\[ \mathbf{q} = \hat{s} \times \mathbf{v} \hat{\mathbf{\Theta}} \]
Screw axis

Given a reference frame, a screw axis $\mathbf{S}$ is defined as:

$$\mathbf{S} = [\mathbf{w}]e^{t\mathbf{R}}$$

where either

1. $\|\mathbf{w}\| = 1$
2. $\mathbf{w} = 0$ and $\|\mathbf{v}\| = 1$

or

- $\mathbf{w} \neq 0$ and $\|\mathbf{v}\| = 1$
- translating along $\mathbf{v}$

$v = -\mathbf{w} \times \mathbf{q} + h\mathbf{w}$

$h = 0$ gives pure rotation

Note: $\mathbf{w} + \mathbf{v}$ are overloading.
Exponential Coordinates for Rigid Body Motions
Logarithm of rigid body motions
Summary

• Used transformations to define spatial velocities as **body and spatial twists**
• Introduced **screw motions** and the screw interpretation of a twist
• The **screw axis** $S$ was defined and used to define the **exponential coordinates of homogeneous transformations**
• Connections between last lecture and this lecture:
  • Transformation matrix $T$ is analogous to rotation matrix $R$
  • A screw axis $S$ is analogous to rotation axis $\hat{\omega}$
  • A twist $\mathcal{V}$ can be expressed as $S \dot{\theta}$ and is analogous to angular velocity $\omega = -\hat{\omega} \dot{\theta}$
  • Exponential coordinates $S \theta \in \mathbb{R}^6$ for rigid body motions are analogous to exponential coordinates $\hat{\omega} \theta \in \mathbb{R}^3$