Lecture 14: inverse kinematics II

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Co-Teaching with Prof. Belabbas

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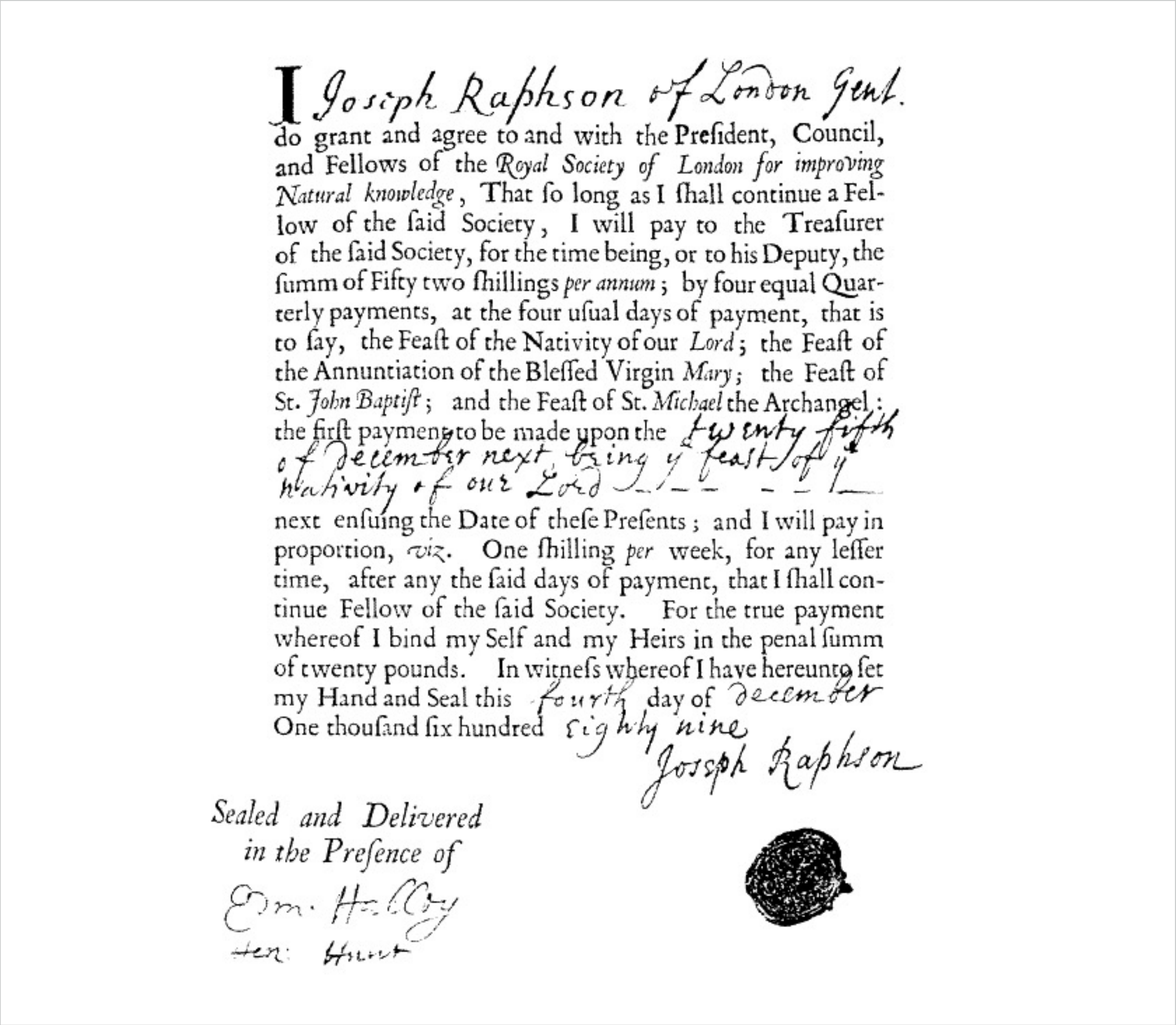
Administrivia

- HW 4 is due ~~next~~ Friday 10/22 at 8pm
- Thursday (10/21) lecture will be replaced with a project catch-up day
- Tuesday (10/26) lecture will be given by Prof. Belabbas
- HW 5 will be posted soon and due Friday 10/29 at 8pm
  - Resources will be posted on discord

Who is Joseph Raphson?

- Joseph Raphson (1648 – 1715) was an English mathematician known best for the **Newton–Raphson method**
- Raphson's most notable work is *Analysis Aequationum Universalis* (1690), which approximates the roots of an equation
- Isaac Newton had developed a very similar formula in his Method of Fluxions, written in 1671, but published in 1736
- Raphson's method is simpler than Newton's, so Raphson's version is generally used in textbooks today



I Joseph Raphson of London Gent.
do grant and agree to and with the Preſident, Council, and Fellows of the *Royal Society of London for improving Natural knowledge*, That ſo long as I ſhall continue a Fellow of the ſaid Society, I will pay to the Treaſurer of the ſaid Society, for the time being, or to his Deputy, the ſumm of Fifty two ſhillings *per annum*; by four equal Quarterly payments, at the four uſual days of payment, that is to ſay, the Feaſt of the Nativity of our *Lord*; the Feaſt of the Annuntiation of the Bleſſed Virgin *Mary*; the Feaſt of St. *John Baptiſt*; and the Feaſt of St. *Michael* the Archangel: the firſt payment to be made upon the twenty fifth of december next being y^e feast of y^e nativity of our Lord next enſuing the Date of theſe Preſents; and I will pay in proportion, *viz.* One ſhilling *per* week, for any leſſer time, after any the ſaid days of payment, that I ſhall continue Fellow of the ſaid Society. For the true payment whereof I bind my Self and my Heirs in the penal ſumm of twenty pounds. In witneſs whereof I have hereunto ſet my Hand and Seal this fourth day of december One thouſand ſix hundred eighty nine

Joseph Raphson

Sealed and Delivered in the Preſence of

Edm. Halley
Hen: Hunt

Credit: Wikipedia

Inverse Kinematics

w

- **Forward Kinematics:** computes the end-effector position from joint angles:

$$\text{joint space} \longrightarrow SE(3)$$
$$\theta \longmapsto T(\theta)$$

- **Inverse Kinematics:** computes the possible joint angles from the pose of the end-effector

$$SE(3) \longrightarrow \text{joint space}$$
$$X \longmapsto \theta$$

- Given $T(\theta)$, find solutions θ that satisfy $T(\theta) = X$
- When the **analytic solution** is hard or impossible to come by, we **numerically** solve $T(\theta) - X = 0$

goal: find roots

Newton-Raphson Methods (1)

Given $g(\theta) = f(\theta) - x_d$, find θ_d such that $g(\theta_d) = 0$

1. Start with initial guess $\theta^{(0)}$ for θ_d
2. Write Taylor expansion of $g(\theta)$ at θ^0 up to the first order

$$g(\theta) = g(\theta^0) + \frac{\partial g}{\partial \theta}(\theta^0)(\theta - \theta^0) + \cancel{h.o.t.} = 0$$

3. Set approximation of g equal to zero and solve for θ

$$g(\theta^0) + \frac{\partial g}{\partial \theta}(\theta^0)(\theta - \theta^0) = 0$$

$$\theta = \theta^0 + \left(\frac{\partial g}{\partial \theta}(\theta^0)\right)^{-1} g(\theta^0)$$

4. Iterate with new estimate of θ_d

$$\theta^{k+1} = \theta^k + \left(\frac{\partial g}{\partial \theta}(\theta^k)\right)^{-1} g(\theta^k)$$

Newton-Raphson Method (2)

$$\theta^{k+1} = \theta^k - \Bigg(\underbrace{\frac{\partial g}{\partial \theta}\Big|_{\theta^k}}_{} \Bigg)^{-1} g(\theta^k)$$

$\partial g / \partial \theta (\theta^k)$

$$\frac{\partial g}{\partial \theta}(\theta) = \begin{bmatrix} \frac{\partial g_1}{\partial \theta_1}(\theta) & \cdots & \frac{\partial g_1}{\partial \theta_n}(\theta) \\ \vdots & \ddots & \vdots \\ \frac{\partial g_n}{\partial \theta_1}(\theta) & \cdots & \frac{\partial g_n}{\partial \theta_n}(\theta) \end{bmatrix} \in \mathbb{R}^{n \times n}$$

we execute this iteration until some stopping condition is met: ex:

$$\frac{|g(\theta^k) - g(\theta^{k+1})|}{|g(\theta^k)|} \leq \varepsilon$$

Newton-Raphson Method (2)

Numerical IK

coordinate vector

Given x_d : end effector position

$f(\theta)$: forward kinematics

θ^0 : initial guess

$g(\theta_d) = x_d - f(\theta_d) = 0$

Find θ_d

$$x_d = f(\theta_d) = f(\theta^0) + \underbrace{\frac{\partial f}{\partial \theta}(\theta^0)}_{J(\theta^0)} \underbrace{(\theta_d - \theta^0)}_{\Delta\theta} + \cancel{\text{hot}}$$

$$\rightarrow x_d - f(\theta^0) = J(\theta^0)\,\Delta\theta$$

if $J(\theta^0)$ is invertible,

$$\Delta\theta = J^{-1}(\theta^0)\,(x_d - f(\theta^0))$$

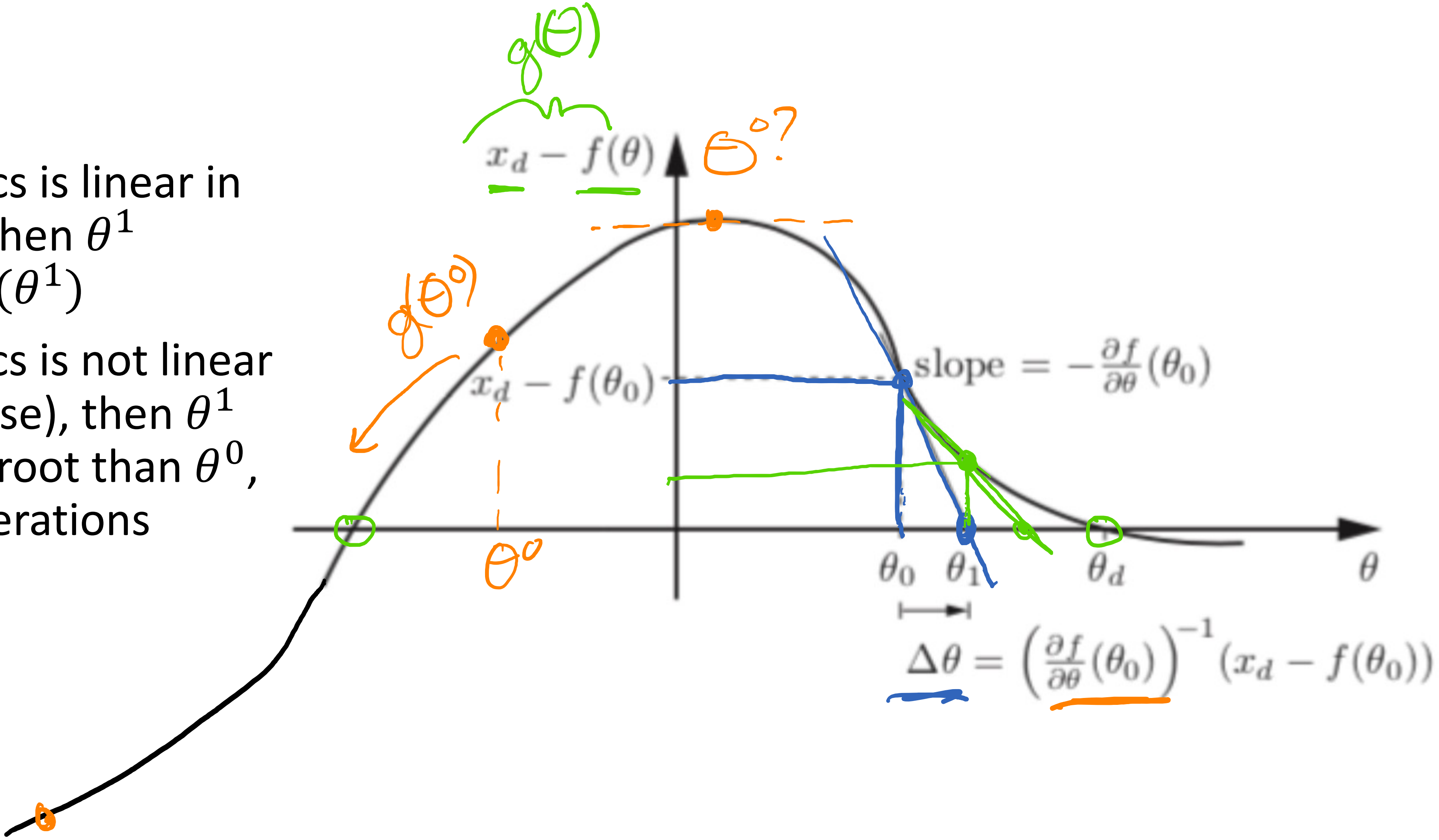
$$\hookrightarrow \theta^1 = \theta^0 + \Delta\theta$$

then reset initial guess θ^0 to θ^1 + iterate:

$\{\theta^0, \theta^1, \theta^2, \ldots\}$ converge to θ_d

Numerical Inverse Kinematics

- If the forward kinematics is linear in θ (i.e., h.o.t. are zero), then θ^1 exactly satisfies $x_d = f(\theta^1)$
- If the forward kinematics is not linear in θ (as is usually the case), then θ^1 should be closer to the root than θ^0, but will require more iterations



Non-invertible Jacobians and the Pseudo-Inverse

- $J(\theta^0)$ may be singular $(\det(J(\theta^0)) = 0)$ or may not be square
  - Non-invertible! Instead use Pseudo-Inverse!

dagger

- The Moore-Penrose pseudo-inverse for $J \in \mathbb{R}^{m\times n}$ is denoted $J^\dagger \in \mathbb{R}^{n\times n}$
- Consider equation: $Jy = z$. We want to find the solution: $y^* = J^\dagger z$, such that:
  - One solution if $n = m$ and J is full rank
    - $y^* = J^{-1}z$
  - Many solutions if $n > m$
    - y^* exactly satisfies $Jy^* = z$, and gives the minimal norm solution ($\|y^*\| \leq \|y\|$)
  - No solutions if $n < m$ and z is not in the span of J
    - If no solution, y^* minimizes the two-norm of the error: $\|Jy^* - z\| \leq \|J\tilde{y} - z\|$ for all $\tilde{y}$

The Pseudo Inverse

- When J is full column rank (n<m, tall):

$$J^+ = \underbrace{(J^T J)^{-1}} J^T$$

(left inverse: $J^+ J = I$) ✓

- When J is full row rank (n>m, wide):

$$J^+ = J^T (J J^T)^{-1}$$

(right inverse: $J J^+ = I$)

- When n=m and full rank:

$$J^+ = J^{-1}$$

$$\Delta\theta = J^+(\theta^0)(x_d - f(\theta^0))$$

Using the Newton-Raphson Method for IK

- The Newton-Raphson algorithm needs to be modified given that $X \in SE(3)$, which is not a general matrix in $\mathbb{R}^{4\times 4}$ or a coordinate vector
- The error vector $e = x_d - f(\theta^i)$, represents the update needed to go from the current guess to the desired end-effector configuration (after being multiplied by the inverse Jacobian)
- Said otherwise, following the direction e for one second, starting from $f(\theta^i)$, should send us (approximately) to x_d
- In our case, we are given $X \in SE(3)$, and instead of computing $X - T(\theta^i)$, we should compute the **twist** $\mathcal{V}_b$ which, if followed for one second, sends us from $T(\theta^i)$ to X

Numerical Inverse Kinematics with a Twist

want to find body twist $\mathcal{V}_b$ to move from $\underbrace{T_{sb}(\theta^i)}_{\text{current pose est}}$ to desired $X_d = \underbrace{T_{sd}}_{\text{desired pose}}$

the twist that sends us from $T_{sb}(\theta^i)$ to T_{sd}

satisfy: $T_{sd} := X = \underbrace{T_{sb}(\theta^i)\, e^{[\mathcal{V}_b]}}$

$$\underbrace{e^{[\mathcal{V}_b]}} = T_{sb}^{-1}(\theta^i)\, T_{sd}$$

recall matrix log:

$$[\mathcal{V}_b] = \log\left(T_{sb}^{-1}(\theta^i) \cdot T_{sd}\right)$$

Numerical Inverse Kinematics: Algorithm

0. Given $X = T_{sd}$ and the forward kinematics map $T_{sb}(\theta)$

   Given tolerances ϵ_ω and ϵ_v

1. Initialize θ^0 and set $i = 0$
2. While $\|\omega_b\| > \epsilon_\omega$ or $\|\omega_v\| > \epsilon_v$ from $\mathcal{V}_b$
   1. Set $[\mathcal{V}_b] = \log T_{sb}^{-1}(\theta^i) T_{sd}$
   2. Set $\theta^{i+1} = \theta^i + J_b^\dagger(\theta^i)\mathcal{V}_b$
   3. Increment i

A few comments

- Re: algorithm on last slide:
  - An equivalent form can be derived in the space frame, using the spatial Jacobian and the spatial twist
- You can also extend this to find sequence of joint angles that will allow the end-effector to follow a trajectory $T_{sb}(t)$
  - Discretize the trajectory $T_{sb}(\theta_k)$ and compute θ_k such that $T_{sb}(\theta_k) = T_{sb}(t_k)$
  - Initialization is *really* important here!

Summary

- **Inverse Kinematics** gives us a method for finding the joint angles given an end-effector configuration
- This can sometimes be done analytically (geometrically), but this is often difficult so **numerical techniques** are more common in practice
- Introduced the **Newton-Raphson method,** which can be modified to solve inverse kinematics numerically
  - In the next HW, you will be given code for the Newton-Raphson method