Introduction to Robotics
Lecture 9: Forward Kinematics: PoE in body frame and Denavit-Hartenberg parameters
Product of exponentials: change of frame

\[ M = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & L_1 + L_2 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix} \]

we expressed the screw vector of each link with respect to frame \( s \) and \( M \) is position of end effector in \( s \).
Recall the change of frame formula for twists: if $S_1$ is the twist of link 1 in frame $s$ and $B_1$ is the twist of link 1 in frame $b$, then

$$[S_1] = T_{sb}[B_1]T_{sb}^{-1} \quad \text{and} \quad [B_1] = T_{bs}[S_1]T_{bs}^{-1}.$$ 

or equivalently,

$$S_1 = \text{Ad}_{T_{sb}} B_1 \quad \text{and} \quad B_1 = \text{Ad}_{T_{bs}} S_1.$$ 

Recall that $M^{-1}e^AM = e^{M^{-1}AM}$. Thus

$$e^AM = Me^{M^{-1}AM}.$$
Recall that in PoE, $M = T_{sb}$ where $s$ is a reference frame and $b$ the end-effector frame. Iterating the previous formula, we get

$$T(\theta) = e^{[S_1 \theta_1]} e^{[S_2 \theta_2]} M$$

$$= e^{[S_1 \theta_1]} M e^{M^{-1}[S_2 \theta_2] M}$$

$$= M e^{M^{-1}[S_1 \theta_1] M} e^{[B_2 \theta_2]}$$

$$= M e^{[B_1 \theta_1]} e^{[B_2 \theta_2]}$$

This is the body-form of the PoE.

Note that we can obtain it from the PoE in space form (i.e. with respect to reference frame), or evaluate directly $B_i$ from the figure.
Product of exponentials: change of frame

$$M = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 3L \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}$$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\omega_i$</th>
<th>$v_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(0, 0, 1)</td>
<td>(0, 0, 0)</td>
</tr>
<tr>
<td>2</td>
<td>(0, 1, 0)</td>
<td>(0, 0, 0)</td>
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<tr>
<td>3</td>
<td>(−1, 0, 0)</td>
<td>(0, 0, 0)</td>
</tr>
<tr>
<td>4</td>
<td>(−1, 0, 0)</td>
<td>(0, 0, L)</td>
</tr>
<tr>
<td>5</td>
<td>(−1, 0, 0)</td>
<td>(0, 0, 2L)</td>
</tr>
<tr>
<td>6</td>
<td>(0, 1, 0)</td>
<td>(0, 0, 0)</td>
</tr>
</tbody>
</table>

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<th>$i$</th>
<th>$\omega_i$</th>
<th>$v_i$</th>
</tr>
</thead>
<tbody>
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<td>1</td>
<td>(0, 0, 1)</td>
<td>(−3L, 0, 0)</td>
</tr>
<tr>
<td>2</td>
<td>(0, 1, 0)</td>
<td>(0, 0, 0)</td>
</tr>
<tr>
<td>3</td>
<td>(−1, 0, 0)</td>
<td>(0, 0, −3L)</td>
</tr>
<tr>
<td>4</td>
<td>(−1, 0, 0)</td>
<td>(0, 0, −2L)</td>
</tr>
<tr>
<td>5</td>
<td>(−1, 0, 0)</td>
<td>(0, 0, −L)</td>
</tr>
<tr>
<td>6</td>
<td>(0, 1, 0)</td>
<td>(0, 0, 0)</td>
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</table>

space-frame

body-frame
Recall the 3R arm. We have $T_{04} = T_{01} \, T_{12} \, T_{23} \, T_{34}$.

The DH formalism provides a set of rules for assigning frames to links.

This formalism makes velocity kinematics easier, and standardizes the way to write forward kinematics.

We will mostly use the PoE formalism in this course, but DH being widely used, we go over the procedure.
• For all links $\hat{z}_i$ is aligned with joint axis $i$ (i.e. with $\hat{s}_i$ of the corresponding screw)

• Assume that $\hat{z}_i$ and $\hat{z}_{i-1}$ do not intersect and are not parallel. Let

$$a_{i-1} = \text{segment intersecting } \hat{z}_{i-1} \& \hat{z}_i, \text{ perpendicular to both.}$$

• Origin of frame $i-1 = \text{intersection of } a_{i-1} \text{ and axis of } \hat{z}_{i-1}$
• Axis $\hat{x}_{i-1}$ is aligned with $a_{i-1}$.
• Axis $\hat{y}_{i-1}$ is obtained using right-hand-rule.

$\rightarrow$ frame $i - 1$ is specified

• To assign $T_i$, we repeat the above with knowledge from joint $i + 1$. 
• Assume frames $i-1$ and $i$ have been specified. We need 4 parameters to obtain $T_{(i-1)i}$.
  1. The length of $a_{i-1}$, called **link length**. (not length of physical link in general)
  2. Angle $\alpha_{i-1}$ between $\hat{z}_{i-1}$ and $\hat{z}_i$ around $\hat{x}_{i-1}$, called **link twist**.
  3. The distance $d_i$ between intersection of $a_{i-1}$ and $\hat{z}_i$ and origin of frame $i$. This is called the **link offset**.
  4. The angle $\phi_i$ between $\hat{x}_{i-1}$ and $\hat{x}_i$ measured about the $\hat{z}_i$ axis. This is called the **joint angle**
• Recall that $T_{si} = T_{s(i-1)} T_{(i-1)i}$. We have

$$T_{(i-1)i} = Rot(\hat{x}, \alpha_{i-1}) Tr(\hat{x}, a_{i-1}) Tr(\hat{z}, d_i) Rot(\hat{z}, \phi_i)$$

$$= \begin{bmatrix}
\cos \phi_i & -\sin \phi_i & 0 & a_{i-1} \\
\sin \phi_i \cos \alpha_{i-1} & \cos \phi_i \cos \alpha_{i-1} & -\sin \alpha_{i-1} & -d_i \sin \alpha_{i-1} \\
\sin \phi_i \sin \alpha_{i-1} & \cos \phi_i \sin \alpha_{i-1} & \cos \alpha_{i-1} & d_i \cos \alpha_{i-1} \\
0 & 0 & 0 & 1
\end{bmatrix}$$
We can visualize $T_{i-1,i}$ as the following sequence of steps:

1. Rotation of frame $i - 1$ about its $\hat{x}$ axis by angle $\alpha_{i-1}$
2. Translation of resulting frame along its $\hat{x}$ axis by distance $a_{i-1}$.
3. Translation of resulting frame along its $\hat{z}$ axis by distance $d_i$.
4. Rotation of the new frame about its $\hat{z}$ axis by angle $\phi_i$. 
DH special cases: intersecting and parallel revolution axes

**Intersecting axes**
- If the two axes of revolution intersect, then \( a_{i-1} = 0 \).
- In this case, set \( \hat{x}_{i-1} \) to be perpendicular to both \( \hat{z}_i \) and \( \hat{z}_{i-1} \).
- Two such \( \hat{x}_{i-1} \) exist, both are fine (they lead to opposite signs for the angle \( \alpha_{i-1} \)).

**Parallel axes**
- If the two axes are parallel, there is an infinite number of choices for the segment \( a_{i-1} \), all of the same length and perpendicular to be \( \hat{z}_i \) and \( \hat{z}_{i-1} \).
- We can choose any of these.
- In practice, choose it so as to make other parameters zero or easy to manipulate, but it is not necessary to do so.
Choose the \( \hat{z} \) direction of the link reference frame to be along positive direction of translation.

- Link offset \( d_i \) is the joint variable and joint angle \( \phi_i \) is constant (opposite situation of revolute joint)

- All else (convention to choose frame origin, choice of \( \hat{x} \) and \( \hat{y} \) axes) remains the same as for revolute joints.
1. length of $a_{i-1} = \text{link length}$
2. $\alpha_{i-1} = \angle \text{b/t } \hat{z}_{i-1} \text{ and } \hat{z}_i \text{ around } \hat{x}_{i-1} = \text{link twist}$.
3. $d_i = \text{dist. b/t } a_{i-1} \cap \hat{z}_i \text{ and orig. frame } i = \text{link offset}$.
4. $\phi_i = \angle \text{b/t } \hat{x}_{i-1} \text{ and } \hat{x}_i \text{ around } \hat{z}_i = \text{joint angle}$

- Frames 1 and 2 are uniquely specified.
- Choose frame 3 so that $\hat{x}_3 = \hat{x}_2$.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\alpha_{i-1}$</th>
<th>$a_{i-1}$</th>
<th>$d_i$</th>
<th>$\phi_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\theta_1$</td>
</tr>
<tr>
<td>2</td>
<td>$90^\circ$</td>
<td>$L_1$</td>
<td>0</td>
<td>$\theta_2 - 90^\circ$</td>
</tr>
<tr>
<td>3</td>
<td>$-90^\circ$</td>
<td>$L_2$</td>
<td>0</td>
<td>$\theta_3$</td>
</tr>
</tbody>
</table>
 DH: RRRP open chain

- $\theta_4$ is displacement of prismatic joint.
- Choose frame 3 so that $\hat{x}_3 = \hat{x}_2$. 
We can easily translate a DH representation into a PoE.

To do so, first recall that for an invertible matrix $M$, $Me^P M^{-1} = e^{MPM^{-1}}$, and thus

$$Me^P = e^{MPM^{-1}} M.$$ 

Recall that

$$T_{(i-1)i} = \text{Rot}(\hat{x}, \alpha_{i-1}) \text{Tr}(\hat{x}, a_{i-1}) \text{Tr}(\hat{z}, d_i) \text{Rot}(\hat{z}, \phi_i).$$

If joint is revolute: set $\theta_i = \phi_i$ and write $\text{Rot}(\hat{z}, \theta_i)$ as the matrix exponential

$$\text{Rot}(\hat{z}, \theta_i) = e^{[A_i] \theta_i}, [A_i] = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$ 

We have $T_{i-1,i} = M_i e^{[A_i] \theta_i}$ with $M_i = \text{Rot}(\hat{x}, \alpha_{i-1}) \text{Tr}(\hat{x}, a_{i-1}) \text{Tr}(\hat{z}, d_i)$. 
• If joint is prismatic: set $\theta_i = d_i$ and write $Tr(\hat{z}, d_i)$ as the matrix exponential

$$Tr(\hat{z}, \theta_i) = e^{[A_i] \theta_i}, [A_i] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We have $T_{i-1,i} = M_i e^{[A_i] \theta_i}$ with $M_i = Rot(\hat{x}, \alpha_{i-1}) Tr(\hat{x}, a_{i-1}) Rot(\hat{z}, \phi_i)$. (Note that the last two operations in $T_{i-1,i}$ commute here!)
• Putting the above together, we have

\[ T_{0,n} = M_1 e^{[A_1]_{\theta_1}} M_2 e^{[A_2]_{\theta_2}} \ldots M_n e^{[A_n]_{\theta_n}} \]

• Using the identity \( Me^P = e^{MPM^{-1}} M \) iteratively, we obtain

\[
T_{0n} = e^{M_1 [A_i]_{M_1^{-1} \theta_1} (M_1 M_2) e^{[A_2]_{\theta_2}} \ldots e^{[A_n]_{\theta_n}} \\
= e^{M_1 [A_i]_{M_1^{-1} \theta_1} e^{(M_1 M_2) [A_2] (M_1 M_2)^{-1} \theta_2} (M_1 M_2 M_3) \ldots e^{[A_n]_{\theta_n}} \\
= e^{[S_1]_{\theta_1}} \ldots e^{[S_n]_{\theta_n}} M
\]

with

\[
[S_i] = (M_1 \ldots M_{i-1}) [A_i] (M_1 \ldots M_{i-1})^{-1} \\
M = M_1 M_2 \ldots M_n\]