Introduction to Robotics
Lecture 8: Forward Kinematics
The forward kinematics of a robot refers to the calculation of the position and orientation of its effector frame from its joint coordinates. Here, simple trigonometry yields

\[ x = L_1 \cos \theta_1 + L_2 \cos(\theta_1 + \theta_2) + L_3 \cos(\theta_1 + \theta_2 + \theta_3) \]

\[ y = L_1 \sin \theta_1 + L_2 \sin(\theta_1 + \theta_2) + L_3 \sin(\theta_1 + \theta_2 + \theta_3) \]
Forward kinematics

- For more complex 3D mechanisms, a direct analysis as done in previous slide is too onerous.
- We describe two systematic, principle ways to perform forward kinematics: Product of Exponentials (PoE) and Denavit-Hartenberg (next lecture).
- What we can do using previous lectures: attaching frames 1,2,3,4 to the three links and end-effector respectively, and denoting by 0 the reference frame, we need $T_{04}$, which we can obtain as

$$T_{04} = T_{01} T_{12} T_{23} T_{34},$$

and $T_{i(i+1)}$ are easy to derive.
Forward kinematics: using homogeneous transformations

\[
T_{01} = \begin{bmatrix}
    \cos \theta_1 & -\sin \theta_1 & 0 & 0 \\
    \sin \theta_1 & \cos \theta_1 & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1
\end{bmatrix},
\quad
T_{12} = \begin{bmatrix}
    \cos \theta_2 & -\sin \theta_2 & 0 & L_1 \\
    \sin \theta_2 & \cos \theta_2 & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1
\end{bmatrix},
\quad
T_{23} = \begin{bmatrix}
    \cos \theta_3 & -\sin \theta_3 & 0 & L_2 \\
    \sin \theta_3 & \cos \theta_3 & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1
\end{bmatrix},
\quad
T_{34} = \begin{bmatrix}
    1 & 0 & 0 & L_3 \\
    0 & 1 & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1
\end{bmatrix}
\]
Assume the arm is in “resting position”:
\[ M := T(I, (L_1 + L_2 + L_3, 0, 0)^\top) \).

From this resting position, what motions are possible?
Revolute joints allow screw motions with zero pitch.
Assume \( \theta_1 = \theta_2 = 0 \). Describe the screw motion of joint 3 by its
twist in frame 0: \( S_3 = [\omega_3, v_3] \).
• How to obtain $\omega_3$, $v_3$?
• Revolute joint: rotation around axis perpendicular to motion and $\omega$ is normalized vector in this direction $\Rightarrow \omega_3 = (0, 0, 1)^\top$.
• To find $v_3$, consider a rigid body attached to the joint and following the joint’s motion: $v_3$ is the velocity of the point at the origin of frame 0. This speed is $\omega \times (-L_1 - L_2, 0, 0)^\top$. 
Forward kinematics: using screw motions

- When only $\theta_3$ is allowed to move, we have
  \[ T_{04} = e^{[S_3]_{\theta_3} M}, \]
  where, by definition,
  \[
  [S_3] = \begin{bmatrix}
  0 & -\omega_3 & \omega_2 & v_1 \\
  \omega_3 & 0 & -\omega_1 & v_2 \\
  -\omega_2 & \omega_1 & 0 & v_3 \\
  0 & 0 & 0 & 0
  \end{bmatrix}.
  \]

- Thus here, $[S_3] = \begin{bmatrix}
  0 & -1 & 0 & 0 & 0 \\
  1 & 0 & 0 & -(L_1 + L_2) & 0 \\
  0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0
  \end{bmatrix}$.
We now repeat the procedure: assume $\theta_1$ fixed at zero, $\theta_3$ fixed at an arbitrary value, and move $\theta_2$. The corresponding twist is

$$S_2 = (\omega_2 = (0, 0, 1)^\top, (0, -L_1, 0))^\top.$$
Forward kinematics: product of exponentials

- Thus \( S_2 \) = 
\[
\begin{bmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & -L_1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

- Now rotation about joint 2 can be viewed as applying a screw motion to the rigid body (link 1 + link 2), thus
\[
T_{04} = e^{[S_2] \theta_2} e^{[S_3] \theta_3} M.
\]
• Finally, $S_1 = ((0, 0, 1)^\top, (0, 0, 0)^\top)$ and

$$T_{04} = e^{[S_1]\theta_1} e^{[S_2]\theta_2} e^{[S_3]\theta_3} M.$$
3R open chain, with non-collinear rotation axes.

Note: \( \hat{z}_i \) aligned with rotation axis, \( \hat{x}_i \) points to next joint.

Forward kinematics as the form

\[
T(\theta) = e^{[S_1] \theta_1} e^{[S_2] \theta_2} e^{[S_3] \theta_3} M.
\]

We need to find \( M \) and the \( S_i \)'s.
- $M$ is the configuration of the end-effector fixed frame (frame 3) when all joint variables are zero.
- We obtain

$$M = \begin{bmatrix} 0 & 0 & 1 & L_1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$
● The screw axis for joint 1 in frame 0 is $S_1 = (\omega_1, v_1)$ with $\omega_1 = (0, 0, 1)^\top$ and $v_1 = (0, 0, 0)^\top$.

● The screw axis for joint 2 in frame 0 is $S_2 = (\omega_2, v_2)$ with $\omega_2 = (0, -1, 0)^\top$ and $v_2 = (0, 0, -L_1)^\top$.

● To obtain $v_2$, set $q$ to be the vector joining origin of reference frame to center of joint, here $q = (L_1, 0, 0)^\top$ and then $v_2 = \omega_2 \times (-q)$.

● Finally, $S_3 = (\omega_3, v_3)$ with $\omega_3 = (1, 0, 0)^\top$ and $v_3 = \omega_3 \times (-L_1, 0, -L_2)^\top) = (0, -L_2, 0)$
Product of exponentials: example: RRPRRRR open chain

Forward kinematics is described by

\[
M = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & L_1 + L_2 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

<table>
<thead>
<tr>
<th>(i)</th>
<th>(\omega_i)</th>
<th>(v_i)</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>(0, 0, 1)</td>
<td>(0, 0, 0)</td>
</tr>
<tr>
<td>2</td>
<td>(1, 0, 0)</td>
<td>(0, 0, 0)</td>
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<td>3</td>
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<td>(0, 1, 0)</td>
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<tr>
<td>4</td>
<td>(0, 1, 0)</td>
<td>(0, 0, 0)</td>
</tr>
<tr>
<td>5</td>
<td>(1, 0, 0)</td>
<td>(0, 0, -L_1)</td>
</tr>
<tr>
<td>6</td>
<td>(0, 1, 0)</td>
<td>(0, 0, 0)</td>
</tr>
</tbody>
</table>