The First Friendly Robot

• In 1923, Prof. Makoto Nishimura saw R.U.R., who became deeply troubled with the idea that humans would struggle against artificial life
  • He was likely influenced by the increasingly popular automata, mechanical figures that would exhibit autonomous behaviors
  • He feared that humanity would be “destroyed by the pinnacle of its creation”

• Nishimura decided to create a different kind of artificial human that would celebrate nature, called Gakutensoku
  • Nishimura refused to call his creation a robot. He named it Gakutensoku, meaning “learning from the rules of nature”

• Disappeared after touring through Japan, China, and Korea

• Gakutensoku had a longstanding impact on Japanese robotics: many robot makers believe that machines are not in opposition to nature but rather part of it

Administrivia

• Homework 2 is due Friday 9/10 at 8pm
• Project Update 0 will be due 9/11 at midnight

on PrairieLearn

⇒ on Gradescope
Angular Velocities

\[ \hat{x}(t + \Delta t) \approx x(t) + \omega \Delta \theta \times \hat{x} \]

as \( \Delta t \to 0 \), \( \dot{x} = \omega \times x \)
Angular Velocities in Reference Frame

Given: \( R(t) \) is orientation of body w.r.t. \( \mathbb{R}^3 \) at \( t \)
\( R(t) = \begin{bmatrix} r_1(t) & r_2(t) & r_3(t) \end{bmatrix} \), \( w_3 = w \) is angular vel
\( \dot{r}_i = w_3 \times r_i \quad \Rightarrow \quad \dot{R} = \begin{bmatrix} w_3 \times r_1 & w_3 \times r_2 & w_3 \times r_3 \end{bmatrix} = w_3 \times R \)
for each \( w \), define a unique skew-symmetric matrix:
\( w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix} \quad \Rightarrow \quad [w] 
\[ A = -A^T \]
\[ \epsilon \in so(3) \]
so: \( \dot{R} = w \times R = [w] R \quad \Rightarrow \quad [w] = \dot{R} R^{-1} \)
Some useful properties and relations

Given any $\omega \in \mathbb{R}^3$ and $R \in SO(3)$, the following holds: $R[\omega]R^\top = [R\omega]$

Now recall: $[\omega_s] = \dot{R}R^\top$

If $R$ is $R_{sb}$, we have that $\omega_s = R\omega_b \iff \omega_b = R^\top \omega_s$

$[\omega_b] = [R^\top \omega_s] = R^\top(\dot{R}R^\top)R = R^\top \dot{R} = R^{-1}\dot{R}$

This gives us: $[\omega_s] = \dot{R}R^\top$ and $[\omega_b] = R^\top \dot{R}$
Introducing
Exponential Coordinates
Recall the Matrix Exponential

If \( \dot{x}(t) = Ax(t), x(0) = x_0 \), where \( x \in \mathbb{R}^n, A \in \mathbb{R}^{n \times n} \), the solution to the ODE is:

\[
x(t) = e^{At}x_0
\]

The matrix exponential / the exponential function can be expanded:

\[
e^{At} = I + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \ldots
\]

Properties:

1. \( \frac{d}{dt} (e^{At}) = Ae^{At} = e^{At}A \)
2. If \( A = PDP^{-1} \), then \( e^{At} = Pe^{Dt}P^{-1} \)
3. If \( AB = BA \), then \( e^Ae^B = e^Be^A = e^{A+B} \)
4. Invertible: \( (e^{At})^{-1} = e^{-At} \)
Exponential Coordinate Representation

• Instead representing orientation as a rotation matrix, we introduce a three-parameter representation: exponential coordinates

• Recall: \( \hat{\omega} \) rotation axis, \( \theta \) angle of rotation

• Then \( \hat{\omega} \theta \in \mathbb{R}^3 \) gives the exponential coordinate representation

• A new interpretation for a frame coincident with \( \{s\} \):
  • A rotation for 1 second around \( \hat{\omega} \) at angular velocity \( \theta \), then the resulting frame is \( R \)
  • A rotation for \( \theta \) seconds around \( \hat{\omega} \) at angular velocity 1, then the resulting frame is \( R \)
Exponential Coordinates of Rotations

Recall: \[ \dot{p} = \dot{\omega} \times p \]
\[ \dot{p} = [\dot{\omega}] p \rightarrow p(t) = e^{[\dot{\omega}]/\theta} p(0) \]

A few tricks:
1. \([\dot{\omega}] \in \text{so}(3), \text{skew-symm} \rightarrow [\dot{\omega}]^3 = -[\dot{\omega}] \text{ and } [\dot{\omega}]^4 = -[\dot{\omega}]^2\)
2. \(\sin x = x - \frac{1}{3!} x^3 + \cdots \text{ and } \cos x = 1 - \frac{1}{2} x^2 + \cdots\)
3. \(e^{-[\dot{\omega}]\theta} = I + [\dot{\omega}]\theta + \frac{1}{2} [\dot{\omega}]^2 \theta^2 + \frac{1}{3!} [\dot{\omega}]^3 \theta^3 + \cdots\)

\(R(\dot{\omega}, \theta) = I + \sin \theta [\dot{\omega}] + (1 - \cos \theta) [\dot{\omega}]^2 = e^{[\dot{\omega}]\theta} \)

→ Rodrigues Formula: \((\dot{\omega}, \theta)\) are exp coordinates of rot matrix \(R\)
Matrix Logarithm

• Given rotation matrix $R$, we need to take the logarithm to find the exponential coordinates:

  \[ \text{exp: } [\hat{\omega}]\theta \in so(3) \rightarrow R \in SO(3) \]

  \[ \text{log: } R \in SO(3) \rightarrow [\hat{\omega}]\theta \in so(3) \]

• If we expand Rodrigues formula:

  \[
  \text{Rot}(\hat{\omega}, \theta) = I + \sin \theta [\hat{\omega}] + (1 - \cos \theta)[\hat{\omega}]^2
  \]

  \[
  \text{Rot}(\hat{\omega}, \theta) = \\
  \begin{bmatrix}
  c_\theta + \hat{\omega}_1^2(1 - c_\theta) & \hat{\omega}_1\hat{\omega}_2(1 - c_\theta) - \hat{\omega}_3s_\theta & \hat{\omega}_1\hat{\omega}_3(1 - c_\theta) + \hat{\omega}_2s_\theta \\
  \hat{\omega}_1\hat{\omega}_2(1 - c_\theta) + \hat{\omega}_3s_\theta & c_\theta + \hat{\omega}_2^2(1 - c_\theta) & \hat{\omega}_2\hat{\omega}_3(1 - c_\theta) - \hat{\omega}_1s_\theta \\
  \hat{\omega}_1\hat{\omega}_3(1 - c_\theta) - \hat{\omega}_2s_\theta & \hat{\omega}_2\hat{\omega}_3(1 - c_\theta) + \hat{\omega}_1s_\theta & c_\theta + \hat{\omega}_3^2(1 - c_\theta)
  \end{bmatrix}
  \]
Matrix Logarithm Method

• If we take the trace of the matrix, we can solve for $\theta$:
  • $\text{tr}(R) := r_{11} + r_{22} + r_{33} = 1 + 2 \cos \theta$

• If we compute $R^\top - R$, we get:
  • $r_{32} - r_{23} = 2\hat{\omega}_1 \sin \theta$
  • $r_{13} - r_{31} = 2\hat{\omega}_2 \sin \theta$
  • $r_{21} - r_{12} = 2\hat{\omega}_3 \sin \theta$

  \[
  [\hat{\omega}] = \frac{1}{2 \sin \theta} (R^\top - R)
  \]

• But what if $\sin \theta = 0$? (called singularities)
  • If $\theta = 2k\pi$, we have rotated by 360 degrees
  • If $\theta = (2k + 1)\pi$, then Rodrigues formula is $R = I + 2[\hat{\omega}]^2$, which gives:
    $\hat{\omega}_i = \pm \sqrt{\frac{r_{ii} + 1}{2}}$ and $2\hat{\omega}_i \hat{\omega}_j = r_{ij}$ for $i \neq j$
Homogeneous Transformations: $\text{SE}(3)$

The special Euclidean group $\text{SE}(3)$ is the set of $4 \times 4$ matrices of the form:

$$T = T(R, p) = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$$

- The inverse of $T$ is $T^{-1} = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix} \in \text{SE}(3)$
- If $T_1$ and $T_2 \in \text{SE}(3)$, then $T_1 T_2 \in \text{SE}(3)$
- $T$ satisfies: $\|Tx - Ty\| = \|x - y\|$ and $(Tx - Tz)^T (Ty - Tz) = (x - z)^T (y - z)$
Representing a configuration with SE(3)

Each frame can represent a body frame in a multi-link mechanism

As before:

\[ T_{ab} T_{bc} = T_{ac} \]
\[ T_{ab} \mathbf{v}_b = \mathbf{v}_a \]
Displacing Frames (1)

Consider frame $T_{sb}$, rotation $\text{Rot}(\hat{\omega}, \theta)$, and translation $\text{Trans}(p)$.

\[
\text{Rot}(\hat{\omega}, \theta) = \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{Trans}(p) = \begin{bmatrix} I & P \\ 0 & 1 \end{bmatrix}
\]

\[
T(R, p) = \begin{bmatrix} R & P \\ 0 & 1 \end{bmatrix} = \text{Trans}(p) \text{Rot}(\hat{\omega}, \theta)
\]

1. If $\hat{\omega}$ and $p$ are in fixed frame:

\[
T_{sb} = T(R, p) T_{sb} = \text{Trans}(p) \text{Rot}(\hat{\omega}, \theta) T_{sb}
\]

\[
= \begin{bmatrix} R & P \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_{sb} & P_{sb} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} RR_{sb} & RP_{sb} + P \\ 0 & 1 \end{bmatrix}
\]
Example Displacement (1)

\[ \omega = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \theta = 90^\circ \]

\[ P = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \]

\[ T_{sb} \sim T_{sb'} \]
Displacing Frames (2)

\[ T(R, p) = \text{Trans}(p) \text{Rot}(\hat{\omega}, \theta) \]

2. If \( \hat{\omega} \) and \( p \) are in body frame:

\[ T_{sb''} = T_{sb} T(R, p) = T_{sb} \text{Trans}(p) \text{Rot}(\hat{\omega}, \theta) \]

\[ = \begin{bmatrix} R_{sb}R & R_{sb}p + p_{sb} \end{bmatrix} \]
Example Displacement (2)

\[ \omega = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \theta = 90^\circ \]

\[ \rho = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \]
Mobile arm example

- Robot arm mounted on wheeled platform. Camera fixed to ceiling.
- \{b\} is body frame, \{c\} end-effector frame, \{e\} frame of object, and \{a\} is fixed frame.
- We assume the camera position and orientation in \{a\} is given.
- From camera measurements, you can evaluate the position and orientation of the body and the object in the camera frame.
- Since we designed our robot and have joint-angle estimates, we can obtain the end-effector position and orientation in the body frame.
- To pick up the object, we need the object position and orientation in the frame of our end-effector.
Summary

• Introduced **exponential coordinates** that allow us to parameterize rotations by the rotation axis $\hat{\omega}$ and the angle of rotation $\theta$

• Walked through the **matrix logarithm** method, which tells us how to recover the exponential coordinates from a rotation matrix

• Started discussing **homogeneous transformations** where we not only look at rotations, but also translations in a single operation