Lecture 05: Exponential Coordinates & Homogeneous Transformations

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The First Friendly Robot

- In 1923, Prof. Makoto Nishimura saw R.U.R., who became deeply troubled with the idea that humans would struggle against artificial life
  - He was likely influenced by the increasingly popular automata, mechanical figures that would exhibit autonomous behaviors
  - He feared that humanity would be “destroyed by the pinnacle of its creation”
- Nishimura decided to create a different kind of artificial human that would celebrate nature, called Gakutensoku
  - Nishimura refused to call his creation a robot. He named it Gakutensoku, meaning “learning from the rules of nature”
- Disappeared after touring through Japan, China, and Korea
- Gakutensoku had a longstanding impact on Japanese robotics: many robot makers believe that machines are not in opposition to nature but rather part of it

This enlightened deity would be revealed “seated on a gilded pedestal with its eyes closed, seemingly lost in thought. In its left hand, it holds an electric light with a crystal-shaped bulb, which it slowly lifts into the air. … The lightbulb illuminates, the giant opens its eyes, as if having a realization. Seemingly pleased with its eureka moment, it smiles. Soon, it shifts its attention to a blank piece of paper lying in front of it and begins to write, eagerly recording its newfound insights.”

https://spectrum.ieee.org/robotics/humanoids/the-short-strange-life-of-the-first-friendly-robot

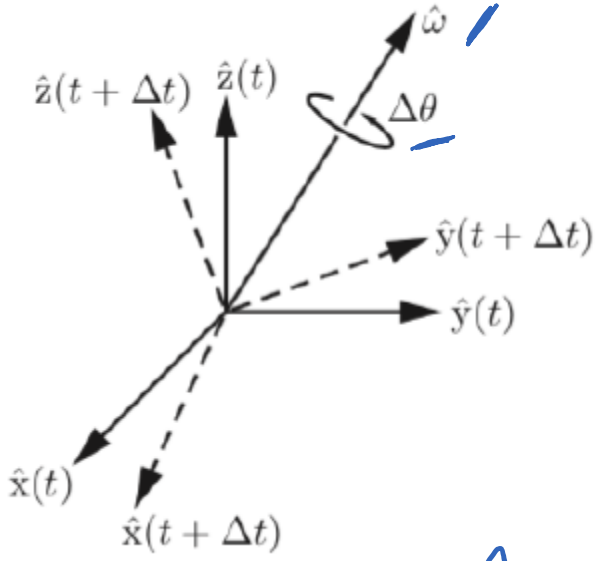
Administrivia

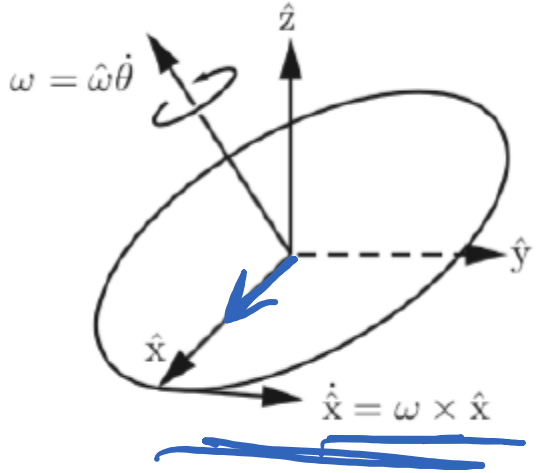
on PrairieLearn

- Homework 2 is due Friday 9/10 at 8pm
- Project Update 0 will be due 9/11 at midnight

↳ on Gradescope

Angular Velocities





$$\hat{x}(t+\Delta t) \simeq \hat{x}(t) + \hat{\omega}\theta \times \hat{x}$$

as $\Delta t \to 0$, $\dot{\hat{x}} = \omega \times \hat{x}$

Angular Velocities in Reference Frame

Given: $R(t)$ is orientation of body w.r.t. $\{s\}$ at t

$R(t) = [r_1(t) \;\; r_2(t) \;\; r_3(t)]$, $\omega_s = \omega$ is angular vel in $\{s\}$!

$\forall i$

$\dot{r}_i = \omega_s \times r_i \rightsquigarrow \dot{R}\,[\underbrace{\omega_s \times r_1 \;\; \omega_s \times r_2 \;\; \omega_s \times r_3}] = \underbrace{\omega_s \times}\, R$

for each ω, define a unique skew-symmetric matrix:

$A = -A^T$

$\in so(3)$

$$\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \longleftrightarrow \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} := [\omega]$$

so: $\dot{R} = \omega \times R = \underbrace{[\omega] R} \longleftrightarrow \underbrace{[\omega]} = \dot{R} R^{-1}$

$= \dot{R} R^T$ ✓

Some useful properties and relations

Given any $\omega \in \mathbb{R}^3$ and $R \in SO(3)$, the following holds: $\underbrace{R[\overset{\mathbb{R}^3}{\omega}]R^\top}_{SO(3)\ 3\times 3} = \underbrace{[\overset{\mathbb{R}^3}{R\omega}]}_{SO(3)\ 3\times 3}$

Now recall: $[\omega_s] = \dot{R}R^\top$

If R is R_{sb}, we have that $\omega_s = R\omega_b \Leftrightarrow \omega_b = R^\top\omega_s$

$$[\omega_b] = [R^\top\omega_s] = R^\top(\dot{R}R^\top)R = R^\top\dot{R} = R^{-1}\dot{R}$$

This gives us: $[\omega_s] = \dot{R}R^\top$ and $[\omega_b] = R^\top\dot{R}$

Introducing Exponential Coordinates

Recall the Matrix Exponential

If $\dot{x}(t) = Ax(t), x(0) = x_0$, where $x \in \mathbb{R}^n, A \in \mathbb{R}^{n\times n}$, the solution to the ODE is:

$$\boxed{x(t) = e^{At}x_0}$$

The matrix exponential / the exponential function can be expanded:

$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots$$

Properties:

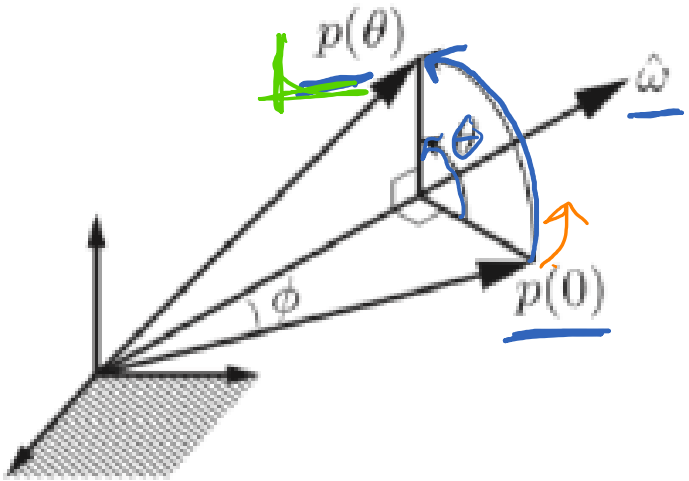
1. $\frac{d}{dt}(e^{At}) = Ae^{At} = e^{At}A$
2. If $A = PDP^{-1}$, then $e^{At} = Pe^{Dt}P^{-1}$
3. If $AB = BA$, then $e^Ae^B = e^Be^A = e^{A+B}$
4. Invertible: $(e^{At})^{-1} = e^{-At}$

Exponential Coordinate Representation

- Instead representing orientation as a rotation matrix, we introduce a three-parameter representation: **exponential coordinates**
- Recall: $\hat{\omega}$ rotation axis, θ angle of rotation
- Then $\hat{\omega}\theta \in \mathbb{R}^3$ gives the exponential coordinate representation

- A new interpretation for a frame coincident with $\{s\}$:
  - A rotation for 1 second around $\hat{\omega}$ at angular velocity θ, then the resulting frame is R
  - A rotation for θ seconds around $\hat{\omega}$ at angular velocity 1, then the resulting frame is R

Exponential Coordinates of Rotations



recall: $\dot{p} = \hat{\omega} \times p$

$$\dot{p} = [\hat{\omega}]p \rightarrow p(\underset{\theta}{t}) = \underbrace{e^{[\hat{\omega}]t}}_{\text{rotation}} p(0) \qquad t \leftarrow \theta$$

a few tricks:

① $[\hat{\omega}] \in so(3)$, skew-symm $\rightarrow [\hat{\omega}]^3 = -[\hat{\omega}]$ and $[\hat{\omega}]^4 = -[\hat{\omega}]^2$

② $\sin x = x - \frac{1}{3!}x^3 + \cdots$ and $\cos x = 1 - \frac{1}{2}x^2 + \cdots$

③ $$\underbrace{e^{[\hat{\omega}]\theta}} = I + [\hat{\omega}]\theta + \frac{1}{2}[\hat{\omega}]^2\theta^2 + \frac{1}{3!}[\hat{\omega}]^3\theta^3 + \cdots$$

$$= I + \underbrace{\left(\theta - \theta^3/3! + \cdots\right)}_{\sin\theta}[\hat{\omega}] + \underbrace{\left(\theta^2/2 - \theta^4/4! + \cdots\right)}_{1-\cos\theta}[\hat{\omega}]^2$$

$$R(\hat{\omega},\theta) = I + \sin\theta[\hat{\omega}] + (1-\cos\theta)[\hat{\omega}]^2 = e^{[\hat{\omega}]\theta}$$

$\rightarrow$ Rodrigues Formula: $(\hat{\omega},\theta)$ are exp coordinates of rot matrix R

Matrix Logarithm

- Given rotation matrix R, we need to take the *logarithm* to find the exponential coordinates:

$$R = \begin{bmatrix} r_{11} & r_{21} & r_{31} \\ r_{12} & \cdot & \cdot \\ r_{13} & \cdot & \cdot \end{bmatrix}$$

exp: $[\hat{\omega}]\theta \in so(3) \rightarrow R \in SO(3)$

log: $\boxed{R \in SO(3)} \rightarrow [\hat{\omega}]\theta \in so(3)$

- If we expand Rodrigues formula:

$$\mathrm{Rot}(\hat{\omega}, \theta) = I + \underbrace{\sin\theta}_{=s_\theta}\,[\hat{\omega}] + \underbrace{(1 - \cos\theta)}_{=c_\theta}[\hat{\omega}]^2$$

$$\mathrm{Rot}(\hat{\omega}, \theta) = \begin{bmatrix} \underbrace{c_\theta + \hat{\omega}_1^2(1 - c_\theta)}_{=r_{11}} & \hat{\omega}_1\hat{\omega}_2(1 - c_\theta) - \hat{\omega}_3 s_\theta & \hat{\omega}_1\hat{\omega}_3(1 - c_\theta) + \hat{\omega}_2 s_\theta \\ \hat{\omega}_1\hat{\omega}_2(1 - c_\theta) + \hat{\omega}_3 s_\theta & c_\theta + \hat{\omega}_2^2(1 - c_\theta) & \hat{\omega}_2\hat{\omega}_3(1 - c_\theta) - \hat{\omega}_1 s_\theta \\ \hat{\omega}_1\hat{\omega}_3(1 - c_\theta) - \hat{\omega}_2 s_\theta & \hat{\omega}_2\hat{\omega}_3(1 - c_\theta) + \hat{\omega}_1 s_\theta & c_\theta + \hat{\omega}_3^2(1 - c_\theta) \end{bmatrix}$$

Matrix Logarithm Method

- If we take the trace of the matrix, we can solve for θ:
  - $tr(R) := r_{11} + r_{22} + r_{33} = 1 + 2\cos\theta$
- If we compute $R^\top - R$, we get:
  - $r_{32} - r_{23} = 2\hat{\omega}_1 \sin\theta$
  - $r_{13} - r_{31} = 2\hat{\omega}_2 \sin\theta$
  - $r_{21} - r_{12} = 2\hat{\omega}_3 \sin\theta$

$$[\hat{\omega}] = \frac{1}{2\sin\theta}(R^\top - R)$$

- But what if $\sin\theta = 0$? (called singularities)
  - If $\theta = 2k\pi$, we have rotated by 360 degrees
  - If $\theta = (2k+1)\pi$, then Rodrigues formula is $R = I + 2[\hat{\omega}]^2$, which gives:

$\hat{\omega}_i = \pm\sqrt{\frac{r_{ii}+1}{2}}$ and $2\hat{\omega}_i\hat{\omega}_j = r_{ij}$ for $i \neq j$

Homogeneous Transformations: SE(3)

The **special Euclidean group** $\boldsymbol{SE(3)}$ is the set of 4×4 matrices of the form:

$$T = T(R, p) = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$$

recall: homogeneous coordinates: if $x \in \mathbb{R}^3 \rightarrow \begin{bmatrix} x \\ 1 \end{bmatrix}$

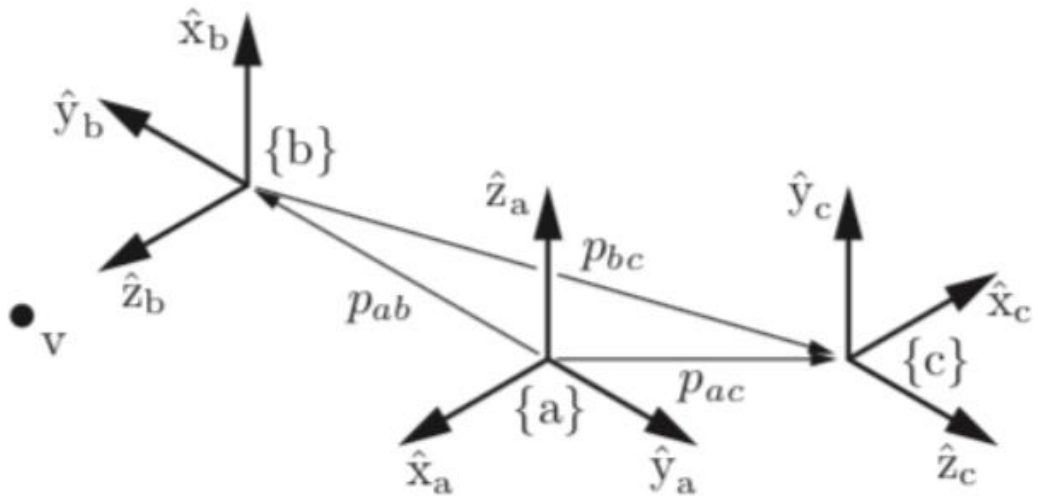
- The inverse of T is $T^{-1} = \begin{bmatrix} R^\top & -R^\top p \\ 0 & 1 \end{bmatrix} \in SE(3)$
- If T_1 and $T_2 \in SE(3)$, then $T_1 T_2 \in SE(3)$
- T satisfies: $\|Tx - Ty\| = \|x - y\|$ and

$$(Tx - Tz)^\top (Ty - Tz) = (x - z)^\top (y - z)$$

rigid motion

- associative
- generally does not commute

Representing a configuration with SE(3)



Each frame can represent a body frame in a multi-link mechanism

As before:

$$T_{ab}T_{bc} = T_{ac}$$

$$T_{ab}v_b = v_a$$

Displacing Frames (1)

consider frame T_{sb}, rotation $Rot(\hat{\omega}, \theta)$, + translation $Trans(p)$

$$Rot(\hat{\omega}, \theta) = \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix}, \quad Trans(p) = \begin{bmatrix} I & p \\ 0 & 1 \end{bmatrix}$$

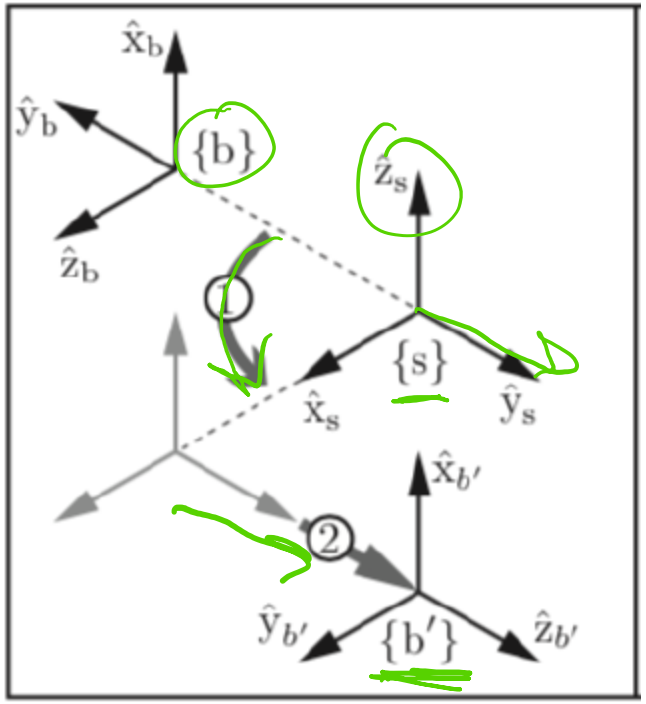
$$T(R, p) = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} = Trans(p)\, Rot(\hat{\omega}, \theta)$$

① if $\hat{\omega}$ and p are in fixed frame:

$$T_{sb'} = T(R, p)\, T_{sb} = Trans(p)\, Rot(\hat{\omega}, \theta) \cdot T_{sb}$$

$$= \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_{sb} & p_{sb} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R R_{sb} & R p_{sb} + p \\ 0 & 1 \end{bmatrix}$$

Example Displacement (1)



$$\hat{\omega} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \theta = 90^\circ$$

$$P = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$T_{sb} \rightsquigarrow T_{sb'}$$

Displacing Frames (2)

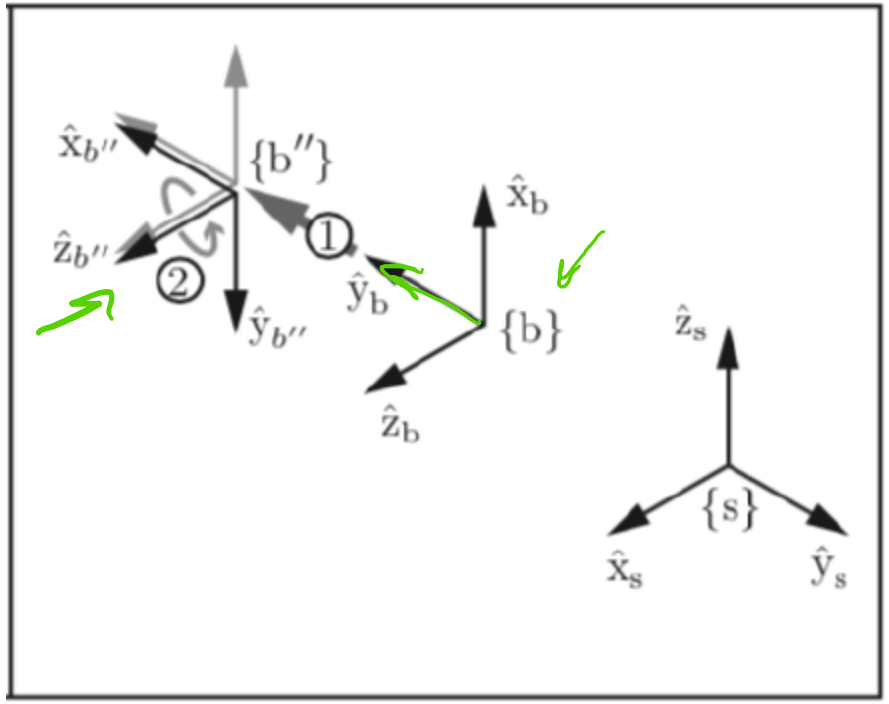
$$T(R,p) = \mathrm{Trans}(p)\,\mathrm{Rot}(\hat{\omega},\theta)$$

② if $\hat{\omega}$ and p are in body frame:

$$T_{sb''} = \underbrace{T_{sb}}\ \underline{T(R,p)} = \underbrace{T_{sb}\,\mathrm{Trans}(p)}\,\mathrm{Rot}(\hat{\omega},\theta)$$

$$= \begin{bmatrix} R_{sb}R & R_{sb}p + p_{sb} \\ 0 & 1 \end{bmatrix}$$

Example Displacement (2)

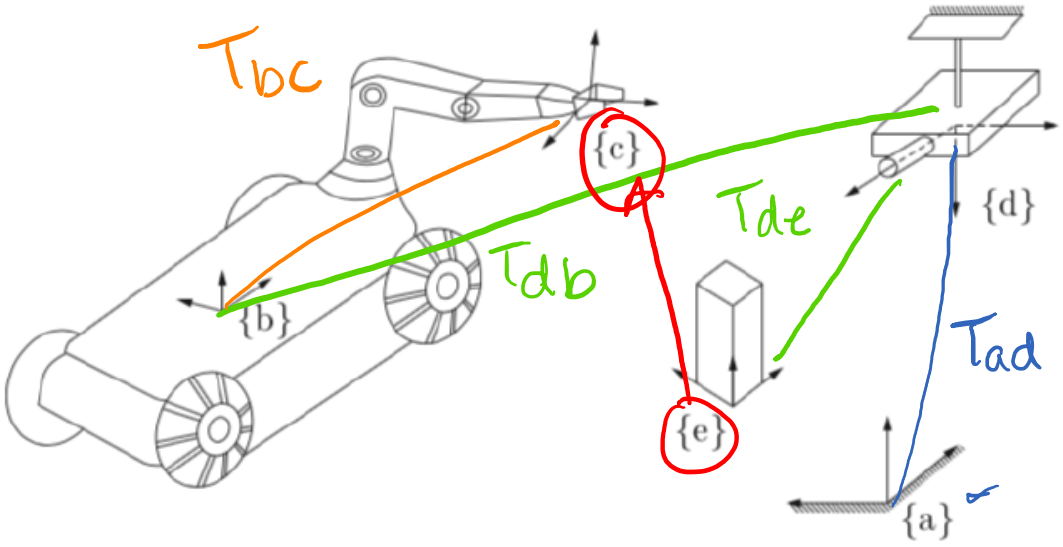


$$\hat{\omega} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \theta = 90^\circ$$

$$p = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

Mobile arm example

- Robot arm mounted on wheeled platform. Camera fixed to ceiling.
- {b} is body frame, {c} end-effector frame, {e} frame of object, and {a} is fixed frame.
- We assume the camera position and orientation in {a} is given.
- From camera measurements, you can evaluate the position and orientation of the body and the object in the camera frame.
- Since we designed our robot and have joint-angle estimates, we can obtain the end-effector position and orientation in the body frame.
- To pick up the object, we need the object position and orientation in the frame of our end-effector.



How to get Tce?

Summary

- Introduced **exponential coordinates** that allow us to parameterize rotations by the rotation axis $\widehat{\omega}$ and the angle of rotation θ
- Walked through the **matrix logarithm** method, which tells us how to recover the exponential coordinates from a rotation matrix
- Started discussing **homogeneous transformations** where we not only look at rotations, but also translations in a single operation