Lecture 04: Constraints cont. & Rigid Body Motions

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Who was Benjamin Olinde Rodrigues?

- A mathematician who turned to finance for employment
  - PhD in math and became a banker
- His dissertation introduced the Rodrigues formula
  - Formerly known as the Ivory–Jacobi formula, as it was independently introduced by Olinde Rodrigues (1816), Sir James Ivory (1824), and Carl Gustav Jacobi (1827)
Administrivia

• Homework 1 is due Friday 9/3 at 8pm
• Project Update 0 will be due 9/11 at midnight
Types of Constraints

Consider some constraints: \( g(\Theta) = \begin{bmatrix} g_1(\Theta_1, \ldots, \Theta_n) \\ \vdots \\ g_k(\Theta_1, \ldots, \Theta_n) \end{bmatrix} = 0 \)

Suppose robot is following traj. \( \Theta(t) \):

\[
\frac{d}{dt} g(\Theta(t)) = \frac{\partial g}{\partial \Theta} (\Theta) \cdot \dot{\Theta} = 0
\]

\[
A(\Theta) \cdot \dot{\Theta} = 0
\]

\( \Rightarrow \) Pfaffian constraints
Types of Constraints

• Suppose we are given Pfaffian constraints: $A(\theta)\dot{\theta} = 0$.

• If we can find a function $g$ such that $\frac{\partial g}{\partial \theta} = A$, the (integrable) constraints $g$ are **holonomic**.
  • Why? If such $g$ exists, the constraints $A(\theta)\dot{\theta} = 0$ are the same as the constraints on the position variables $g(\theta)$.

• If no such $g$ exists, the constraints are called **non-holonomic**.
Rolling Penny Example

A coin rolls w/o slipping:
- Direction given by \((\cos \phi, \sin \phi)\)
- Forward speed is \(r \dot{\phi}\)

\[
\begin{bmatrix}
\dot{x} \\
\dot{y}
\end{bmatrix} = r \dot{\phi} \begin{bmatrix}
\cos \phi \\
\sin \phi
\end{bmatrix}
\]

No slip constraint:
\[
\begin{bmatrix}
1 & 0 & 0 & -rcos \phi \\
0 & 1 & 0 & -rsin \phi
\end{bmatrix} \begin{bmatrix}
\dot{q}_1 \\
\dot{q}_2 \\
\dot{q}_3 \\
\dot{q}_4
\end{bmatrix} = 0 \implies A(q) \dot{q} = 0
\]

Check: \(q\) s.t. \(\partial g / \partial \theta = A\), then \(A_{ij} = \partial g_i / \partial \theta_j\)
Rigid Body Motions
Frames of Reference

Body frame origin $p$ can be expressed:

$$p = \begin{bmatrix} p_x \\ p_y \end{bmatrix} = p_x \hat{x}_s + p_y \hat{y}_s$$

Orientation of $EB^3$ can be written:

$$\hat{x}_b = \cos \theta \hat{x}_s + \sin \theta \hat{y}_s$$

$$\hat{y}_b = -\sin \theta \hat{x}_s + \cos \theta \hat{y}_s$$

To express everything in terms of $ES^3$, we need:

$$p = \begin{bmatrix} p_x \\ p_y \end{bmatrix} + P = \begin{bmatrix} \hat{x}_b \\ \hat{y}_b \end{bmatrix}$$

$$P = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

Rotation matrix
Rotation Matrices (1)

• 2D rotations:
  \[ R(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \]

• 3D rotations about a main axis:
  \[ R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix} \]

Credit: Wikipedia
Rotation Matrices (2)

- We use the convention $R_{ab}$ for the matrix whose **columns are coordinates of the frame** $b$ **expressed in the frame** $a$.

- We have the following laws:
  \[ R_{ab}R_{bc} = R_{ac} \text{ and } R_{ab}^{-1} = R_{ba} \]

- If the vector $p_a$ is the vector $p$ expressed in frame $a$, then
  \[ R_{ba}p_a = p_b \]

- If we rotate a vector $v$ around the $\hat{\omega}$ axis by angle $\theta$, we say $\text{Rot}(\hat{\omega}, \theta)v$.
  - For a rotation about the $x$-axis: $\text{Rot}(\hat{x}, \theta)v = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix}v$
  - Rotations about the $y$ and $z$ axis follow...
Special Orthogonal Groups

\( \textbf{SO}(3) \), the group that represents all spatial rotation matrices, is the set of all \( 3 \times 3 \) real matrices \( R \) that satisfy

1. \( R^\top R = I \)
2. \( \det R = 1 \)

• Note that (2) implies that both (1) holds and that we obey the righthand rule

Similarly, \( \textbf{SO}(2) \), the group describes all planar rotation matrices, is the set of all \( 2 \times 2 \) real matrices \( R \), satisfying the same conditions.
(P, p): parameterized by $\Theta$, describe the position and orientation of $\mathbb{E}_b^3$ relative to $\mathbb{E}_s^3$.

$(R, r)$ gives $\mathbb{E}_c^3$ relative to $\mathbb{E}_s^3$ for $\mathbb{E}_c^3$ in $\mathbb{E}_b^3$ coordinates:

$\mathbf{q} = \begin{bmatrix} q_x \\ q_y \\ q_z \end{bmatrix}$, $\mathbf{Q} = \begin{bmatrix} \cos \psi & -\sin \psi \\ \sin \psi & \cos \psi \end{bmatrix}$

If we know $(\mathbf{Q}, \mathbf{q}) + (P, p)$, we can compute $\mathbb{E}_c^3$ relative to $\mathbb{E}_s^3$:

$R = PQ$, $r = Pq + p$
Frames in 3D

\[ Pa = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad Pb = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad Pc = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \]

1. \( R_a = I \)  
2. Rot about \( \hat{z} \) by 90°.  
3. Rot about \( \hat{y}_b \) by -90°.  
4. \( R_c = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \)
Affine Transformations (more next lecture!)

To rotate and translate in one operation, use homogeneous coordinates:

\[ A = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix}, \quad P = \begin{bmatrix} x \\ 1 \end{bmatrix} \]

\[ A P = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} Rx + t \\ 1 \end{bmatrix} \]
Rigid Body Motions
Recall the cross product

• The cross-product is defined as
  \[ a \times b = ||a|| ||b|| \sin \theta \, \hat{n} \]

• In coordinates:
  \[ a \times b = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix} \]
Angular Velocities

for small rotation $\Delta \Theta$ from $t$ to $t + \Delta t$ of frame $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ about $\hat{\mathbf{\omega}}$, we observe:

$$\mathbf{x}(t + \Delta t) \approx \mathbf{x}(t) + \hat{\mathbf{\omega}} \cdot \Delta \Theta \times \mathbf{x}$$

as $\Delta t \to 0$, $\dot{\mathbf{x}} = \mathbf{w} \times \mathbf{x}$

$\dot{\mathbf{y}} = \mathbf{w} \times \mathbf{y}$

$\dot{\mathbf{z}} = \mathbf{w} \times \mathbf{z}$

$\mathbf{w} = \hat{\mathbf{\omega}} \cdot \Theta$
Angular Velocities in Reference Frame

Let $R(t)$ be orientation of a body wrt $\mathbb{E}^3$ at $t$:

$$R(t) = \begin{bmatrix} r_1(t) & r_2(t) & r_3(t) \end{bmatrix}$$

Let $\omega_3 = \omega$ be the angular vel in $\mathbb{E}^3$:

$$\dot{r}_i(t) = \omega \times r_i \Rightarrow \dot{R} = \begin{bmatrix} \omega \times r_1 & \omega \times r_2 & \omega \times r_3 \end{bmatrix} = \omega \times R$$

For each $\omega$, define a unique skew-symmetric matrix:

$$\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \leftrightarrow \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \omega \end{bmatrix}$$
Some useful properties and relations

Given any $\omega \in \mathbb{R}^3$ and $R \in SO(3)$, the following holds: $R[\omega]R^T = [R\omega]$

Now recall: $[\omega_s] = \dot{R}R^T$

If $R$ is $R_{sb}$, we have that $\omega_s = R\omega_b \iff \omega_b = R^T\omega_s$

$[\omega_b] = [R^T\omega_s] = R^T(\dot{R}R^T)R = R^T\dot{R} = R^{-1}\dot{R}$

This gives us: $[\omega_s] = \dot{R}R^T$ and $[\omega_b] = R^T\dot{R}$
Summary

• Discussed **holonomic and non-holonomic constraints** which may affect the configuration space and mobility of a robot

• Formally defined **frames of reference** and **SO(3)**, and built intuition for rigid body *motion*

• Defined **bracket notation** $[\omega]$, a skew-symmetric representation of our angular velocity