Lecture 03: DoF + Configurations

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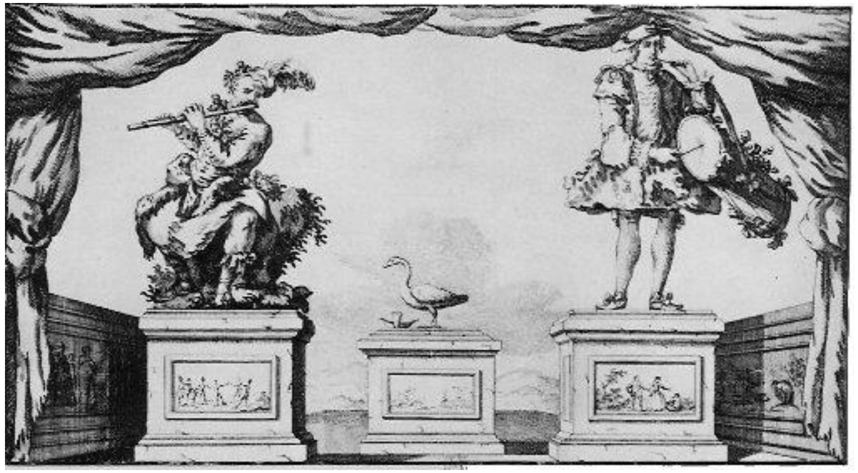
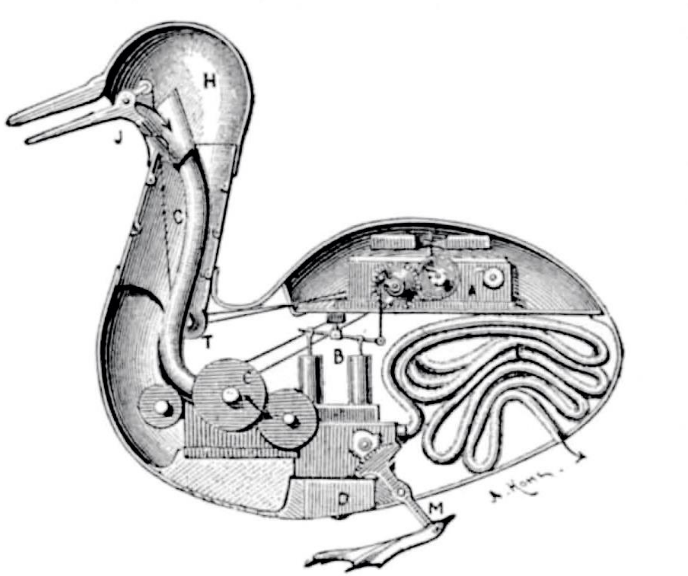
Vaucanson’s Duck

- On 30 May 1739 in France, Jacques de Vaucanson unveiled a mechanical duck
- The duck had over 400 moving parts in each wing alone, and could flap its wings, drink water, “digest” grain, and seemingly defecate
- Robert-Houdin described this as “a piece of artifice I would happily have incorporated in a conjuring trick”
- Voltaire wrote in 1741 that “Without the voice of le Maure and Vaucanson's duck, you would have nothing to remind you of the glory of France."





INTERIOR OF VAUCANSON’S AUTOMATIC DUCK.

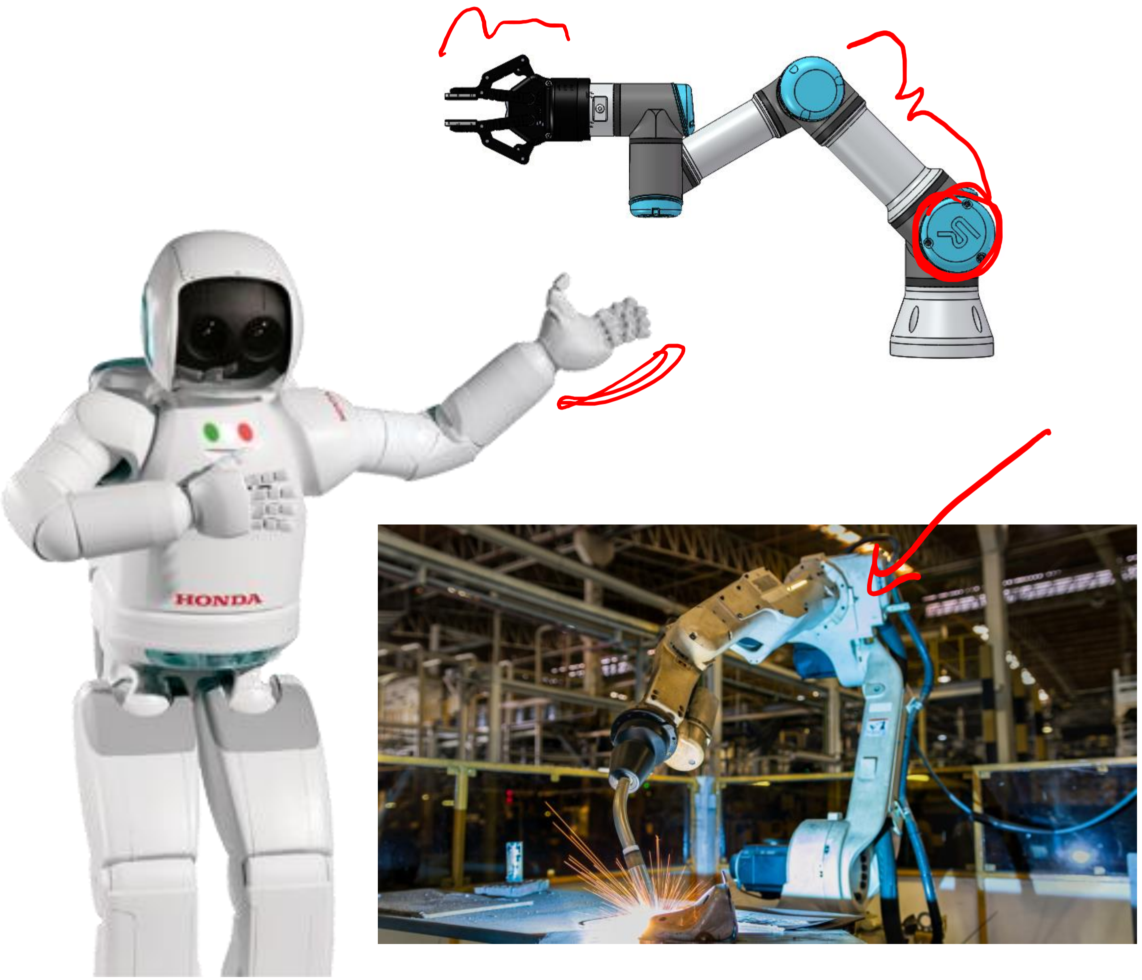
A, clockwork; *B*, pump; *C*, mill for grinding grain; *F*, intestinal tube; *J*, bill; *H*, head; *M*, feet.

Credit: Wikipedia

Administrivia

- Homework 1 is due Friday 9/3 at 8pm
- Office Hours are now posted in Discord (and website ~~soon~~)
- Project Update 0 will be due 9/11 at midnight

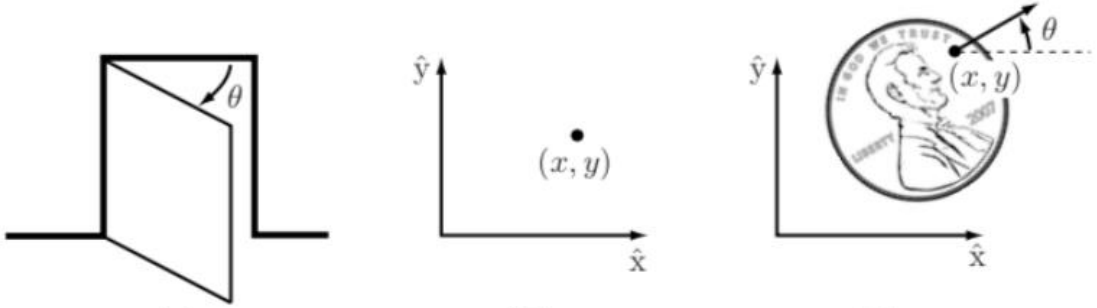
Some Definitions (1)



- A robot is a mechanical device constructed by a set of bodies, called **links**, using **joints**
- A robot moves thanks to **actuators** providing forces and torques
- An **end-effector or tool** (such as a gripper) is attached to a specific link

Some Definitions (2)

- We use our knowledge of the links, joints, and actuators to represent the **configuration** of the robot
- The **configuration space (C-space)** is the n-dimensional space containing all possible configurations of the robot

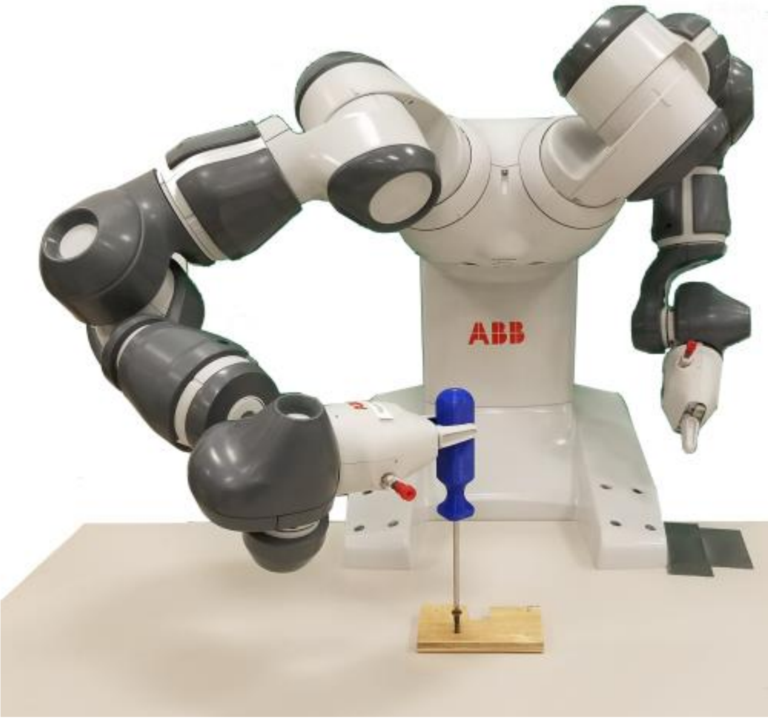


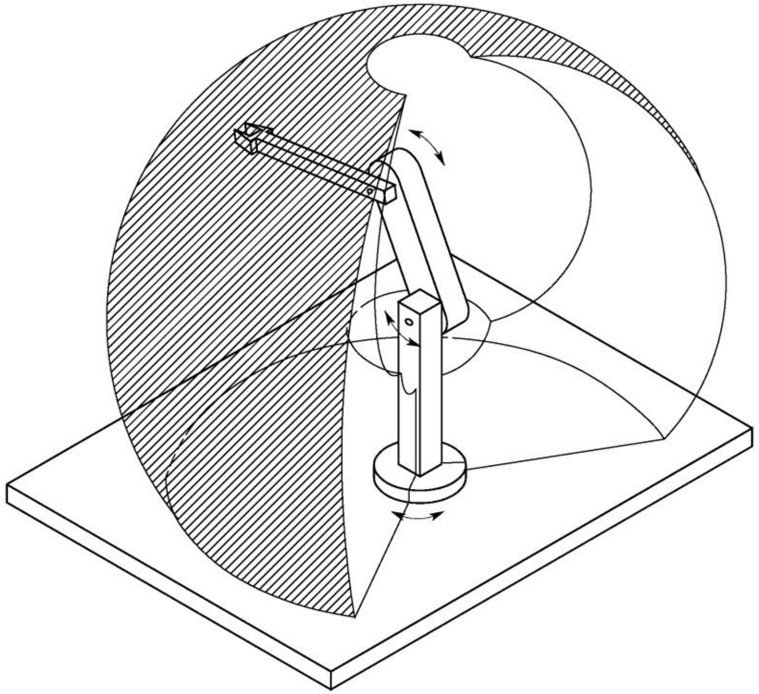
(a) (b) (c)

- The number of **degrees of freedom** of the robot is the smallest number of coordinates needed to represent the configuration

More Definitions (3)

- The **task space** is the space in which the robot's task can be naturally expressed
  - For manipulation, a natural representation is the C-space of the robot's end-effector
  - Driven by the task, independent of the robot
- The **workspace** is the specification of the configurations that the end-effector of the robot can reach
  - Driven by the robot structure, independent of the task
- Both involve choice by the user and are distinct from the robot's C-space





Degrees of Freedom of a Rigid Body

consider a rigid body: a coin!

3 points are sufficient to capture config

DoF = num_var − num_independent_constraints

→ 6 variables to describe 3 points

→ distances (d_{AB}, d_{AC}, d_{BC}) are constraints

DoF = 6 − 3 = 3 → (x, y, θ)

Ind. constraints: $g_i(x_A, y_A, x_B, y_B, x_C, y_C) = 0$

constraints are ind if:

$$\partial g / \partial x = \left(\partial g_i / \partial x_i \right) \text{ is full rank}$$

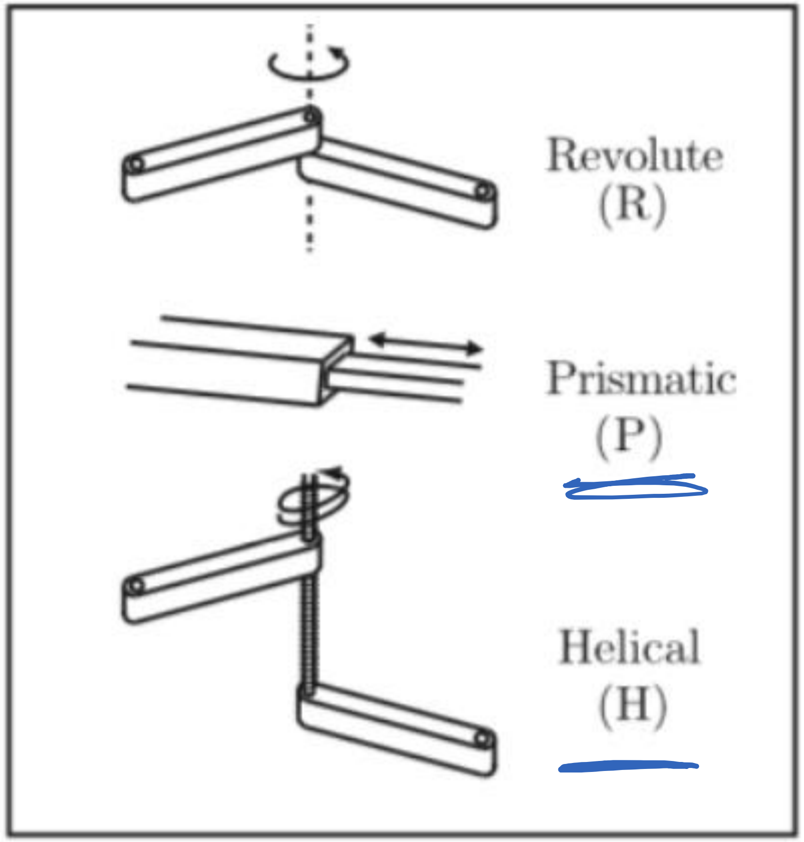
Degrees of Freedom of a Robot

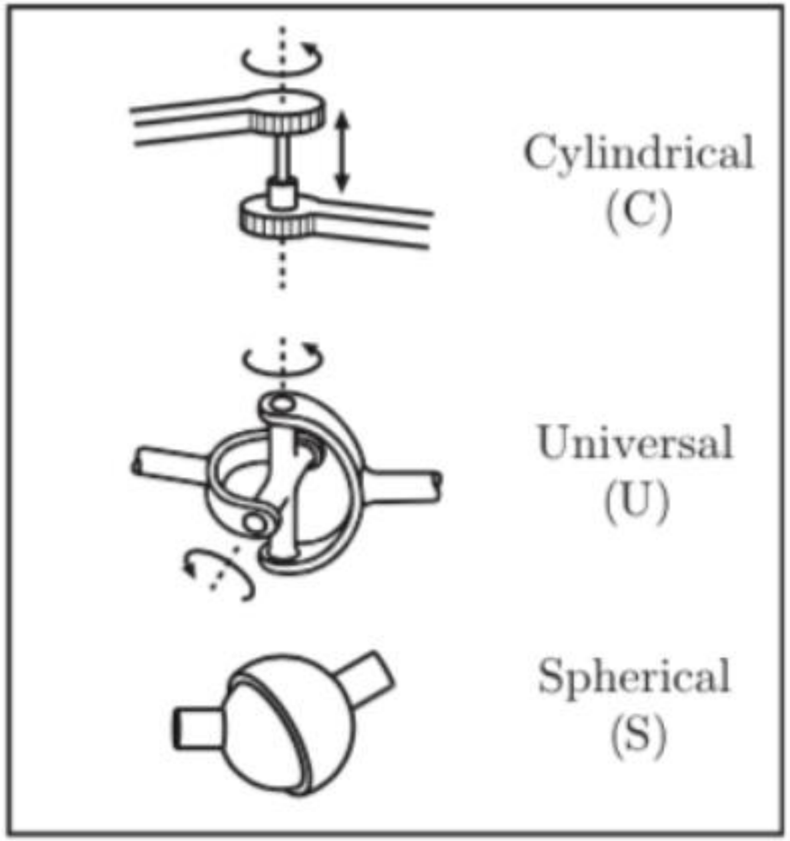
DoF

1

1

1





DoF

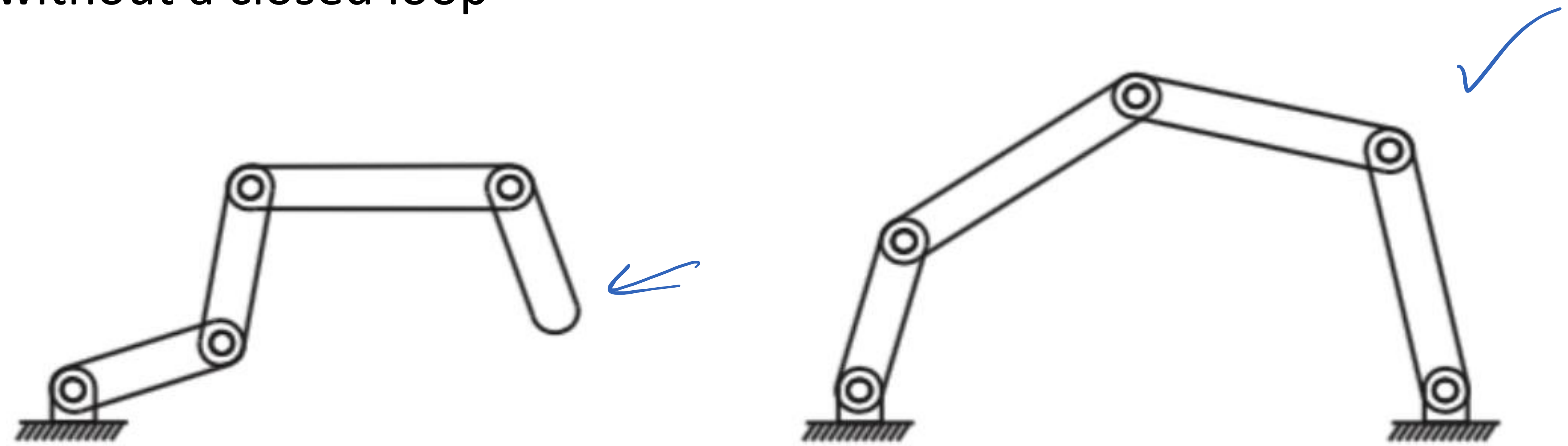
2

2

3

Even More Definitions

- **Closed-chain mechanisms** are mechanisms that have a closed loop between the links and the ground
- **Open-chain mechanisms** (serial mechanisms) is any mechanism without a closed loop



Grübler's Formula

consider a mechanism (robot) with:

- N links $+1 \rightarrow$ ground
- J joints
- m DoF of a rigid body $\rightarrow$ if planar, $m=3$; if spatial, $m=6$
- f_i number of freedons from joint i
- c_i number of constraints for joint i

~~$f_i + c_i = m \ \forall i$~~

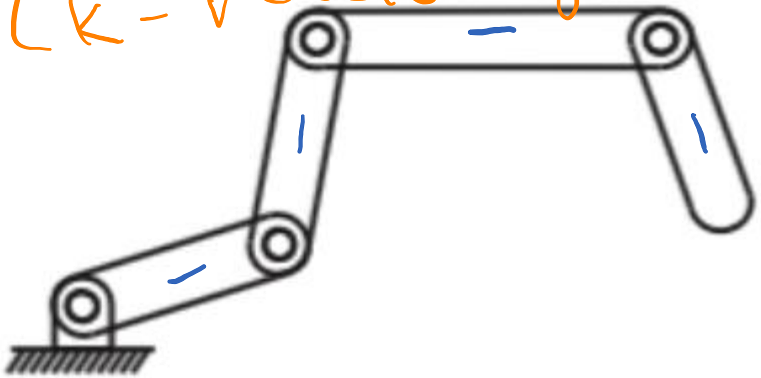
$$f_i + c_i = m \quad \forall i$$

then:

$$\begin{aligned} \text{DoF} &= m(N-1) - \sum_i c_i \\ &= m(N-1) - \sum_i (\underline{m} - f_i) \\ &= m(N-J-1) + \underline{\sum_i}\, \underline{f_i} \ \checkmark \end{aligned}$$

Grubler's Formula – Example

KR robot
(k-revolute joints)



links: $N = k + 1$

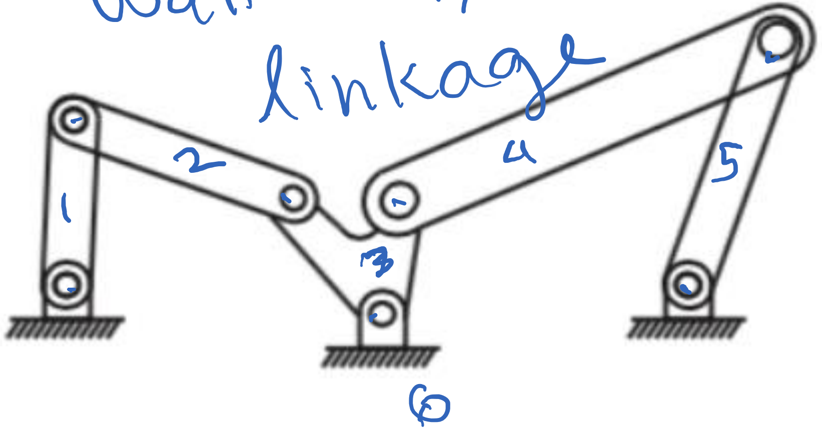
joints: $J = k$

freedoms: $f_i = 1 \quad \forall i$

dof: $m = 3$

$\therefore \text{DoF} = 3\left((k+1) - k - 1\right) + k = k$

Watt six-bar linkage



$N = 6$

$J = 7$

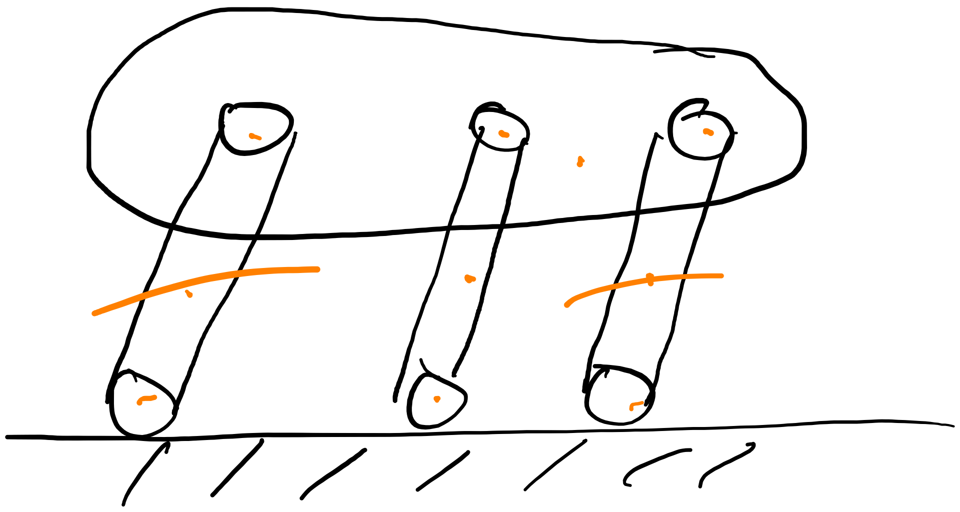
$m = 3$

$f_i = 1 \quad \forall i$

$\therefore \text{DoF} = 3(6 - 7 - 1) + 7 = 1$

Redundant Constraints

parallelogram linkage



$N=5$, $J=6$

$f_i = 1 \ \forall i$ $\quad m=3$

by Grübler:

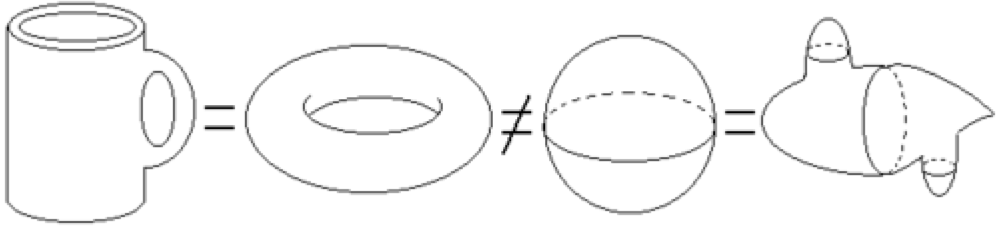
$$\text{DoF} = 3(5-6-1)+6$$
$$= 0$$

"remove" dependent links:

$$\text{DoF} = 3(4-4-1)+4 = 1$$

Configuration Space: Topology

- If dof $= n$, then is the C-space is $\mathbb{R}^n$?
  - No! Spaces can have different shapes or **topologies**
- A circle is written as a S or S^1
- A line can be written as $\mathbb{E}$ or $\mathbb{E}^1$ or $\mathbb{R}$
- A choice of n coordinates to represent an n-dimensional space is called an **explicit parametrization**.



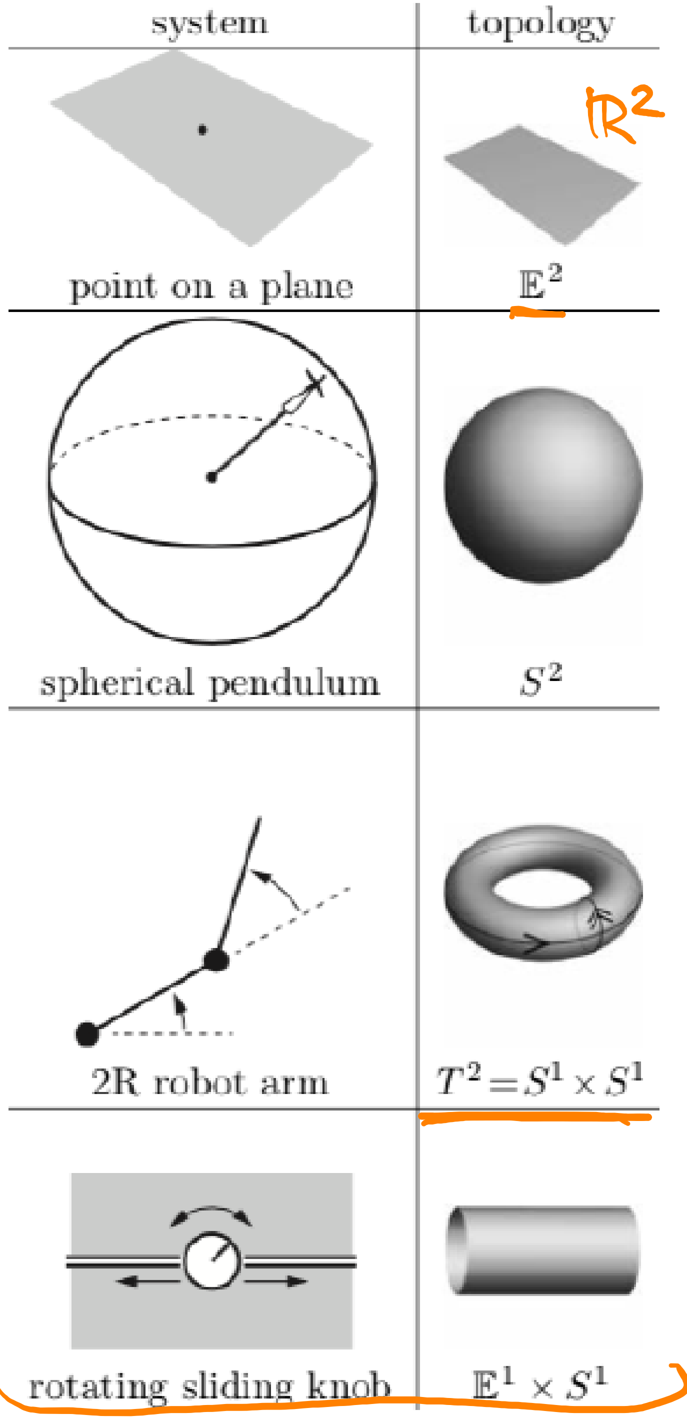


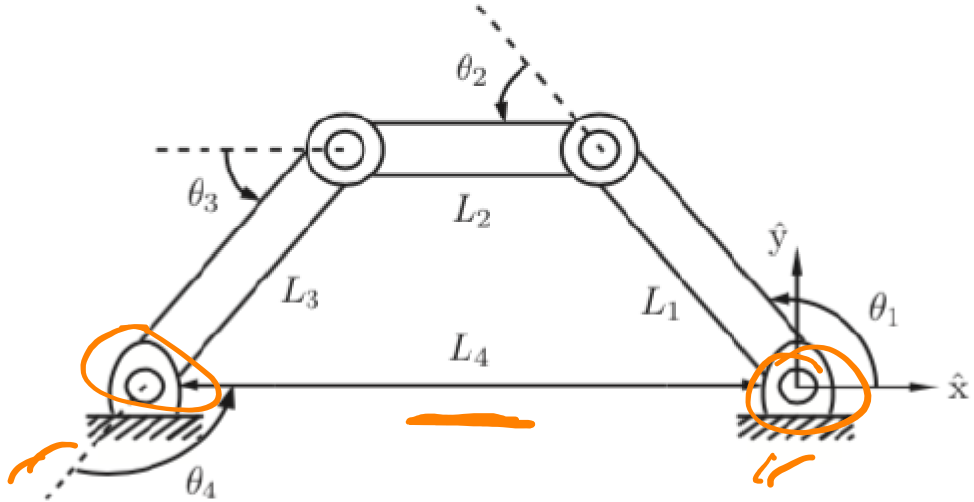
Image Credit: [Wikipedia](#)
and [Renaissance Universal](#)

Configuration Space Representation

- An **implicit representation** is given by constrained coordinates
- It is often easier to obtain than an **explicit representation**
  - For instance, instead of a spherical representation, we can use a higher dimensional space like (x, y, z) and subject it to constraints to reduce the DoF $(x^2 + y^2 + z^2 = 1)$

Configurations and Constraints

How many DoF?



$$
\begin{aligned}
L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) + \cdots + L_4 \cos(\theta_1 + \cdots + \theta_4) &= 0 \\
L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2) + \cdots + L_4 \sin(\theta_1 + \cdots + \theta_4) &= 0 \\
\theta_1 + \theta_2 + \theta_3 + \theta_4 - 2\pi &= 0
\end{aligned}
$$

Types of Constraints

Holonomic constraints decrease the dimension of the C-space, while **non-holonomic** constraints do not.

consider loop closure constraints: $g(\theta) = \begin{bmatrix} g_1(\theta_1, \ldots, \theta_n) \\ \vdots \\ g_k(\theta_1, \ldots, \theta_n) \end{bmatrix} = 0$

suppose the robot is following a time traj. $\theta(t)$

$$\frac{d}{dt} g(\theta(t)) = 0 \rightarrow \begin{bmatrix} \partial g_1/\partial\theta_1(\theta) \cdot \dot{\theta}_1 + \cdots + \partial g_1/\partial\theta_n \cdot \dot{\theta}_n \\ \vdots \\ \partial g_k/\partial\theta_1 \cdot \dot{\theta}_1 + \cdots + \partial g_k/\partial\theta_n \cdot \dot{\theta}_n \end{bmatrix} = 0$$

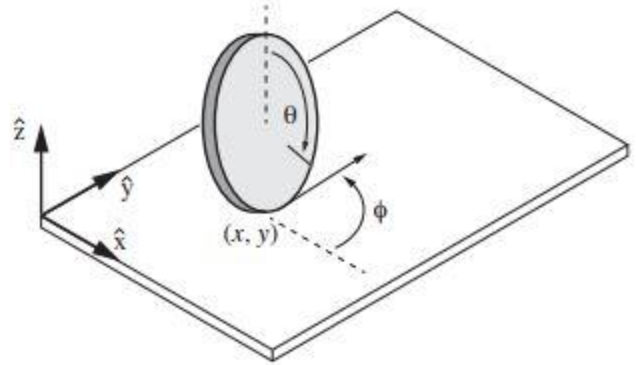
$$= \underbrace{\partial g/\partial\theta(\theta)}_{A(\theta)} \cdot \dot{\theta} = 0$$

$$A(\theta) \cdot \dot{\theta} = 0 \rightarrow \text{Pfaffian}$$

Types of Constraints

- Suppose we are given Pfaffian constraints $A(\theta)\dot{\theta} = 0$
- If we can find a function g such that $\frac{\partial g}{\partial \theta} = A$, the (integrable) constraints g are **holonomic**
  - Why? If such g exists, the constraints $A(\theta)\dot{\theta} = 0$ are the same as the constraints on the position variables $g(\theta)$
- If no such g exists, the constraints are called **non-holonomic**

Rolling Penny Example



Summary

- Introduced fundamental robotics definitions like **configuration space, task space,** and **workspace**
- Learned methods for determining the number of **degrees of freedom** for a rigid body and robot mechanism
- Discussed **holonomic and non-holonomic constraints** which may affect the configuration space and mobility of a robot