lecture 11
forward kinematics II + URDF

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Modern Robotics Ch 4

Admin

- Demo for PF is this week
  - Please start HW3 on PrairieLearn so I can show you your grade
- HW6 due Friday
- Reflection 2 due Sunday 3/8 at midnight
- Please fill out the class feedback form by Sunday 3/8

Denavit-Hartenberg Parameters

- In the 1950s, when Dick Hartenberg, a professor, and Jacques Denavit, a PhD student, developed a way to represent mathematically how mechanisms move
- They showed that the position of one link connected to another could be represented **minimally** using only four parameters
  - Known as the Denavit-Hartenberg (DH) parameters
- In 1981m Richard Paul demonstrated its value for the kinematic analysis of robotic systems

https://robotics.northwestern.edu/history.html

Last Time: Spatial Forward Kinematics

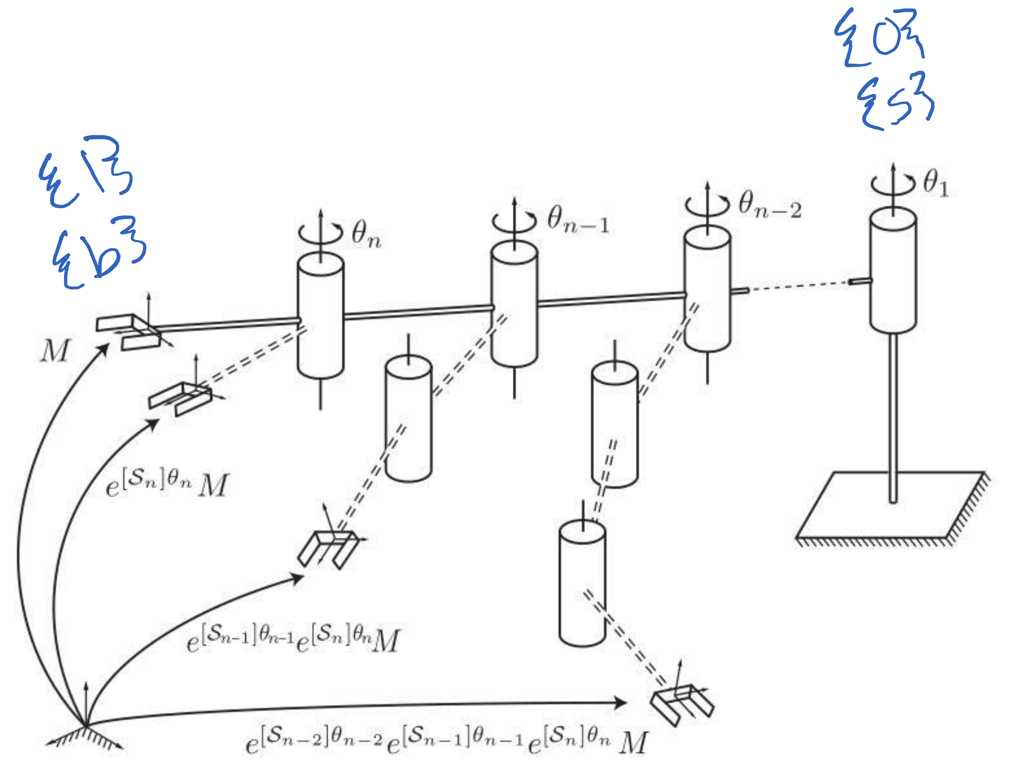


Figure 4.2: Illustration of the PoE formula for an n-link spatial open chain.

- Initialization steps:
  - Choose a fixed frame $\{s\}$
  - Choose an end-effector (tool) frame $\{b\}$
  - Put all joints in *zero position*
  - Let $M \in SE(3)$ be the configuration of $\{b\}$ in the $\{s\}$ frame when the robot is in the zero position
- For each joint i, define the screw axis
- For each motion of a joint, define the screw motion
- These operations compose nicely through multiplication, giving us the Product of Exponentials (PoE) formula!

True form of the Matrix Exponential

$$e^{[\mathcal{S}]\theta} \quad = \quad I + [\mathcal{S}]\theta + [\mathcal{S}]^2\frac{\theta^2}{2!} + [\mathcal{S}]^3\frac{\theta^3}{3!} + \cdots$$

Proposition 3.25. *Let $\mathcal{S} = (\omega, v)$ be a screw axis. If $\|\omega\| = 1$ then, for any distance $\theta \in \mathbb{R}$ traveled along the axis,*

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} e^{[\omega]\theta} & \left(I\theta + (1-\cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2\right) v \\ 0 & 1 \end{bmatrix}. \tag{3.88}$$

If $\omega = 0$ and $\|v\| = 1$, then

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix}.$$

How to compute?

Proposition 3.25. *Let $\mathcal{S} = (\omega, v)$ be a screw axis. If $\|\omega\| = 1$ then, for any distance $\theta \in \mathbb{R}$ traveled along the axis,*

skew symm.

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} \underline{e^{[\omega]\theta}} & \left(I\theta + (1-\cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2\right) v \\ 0 & 1 \end{bmatrix}. \qquad (3.88)$$

Compute $e^{[\omega]\theta}$ just like any matrix exponential!

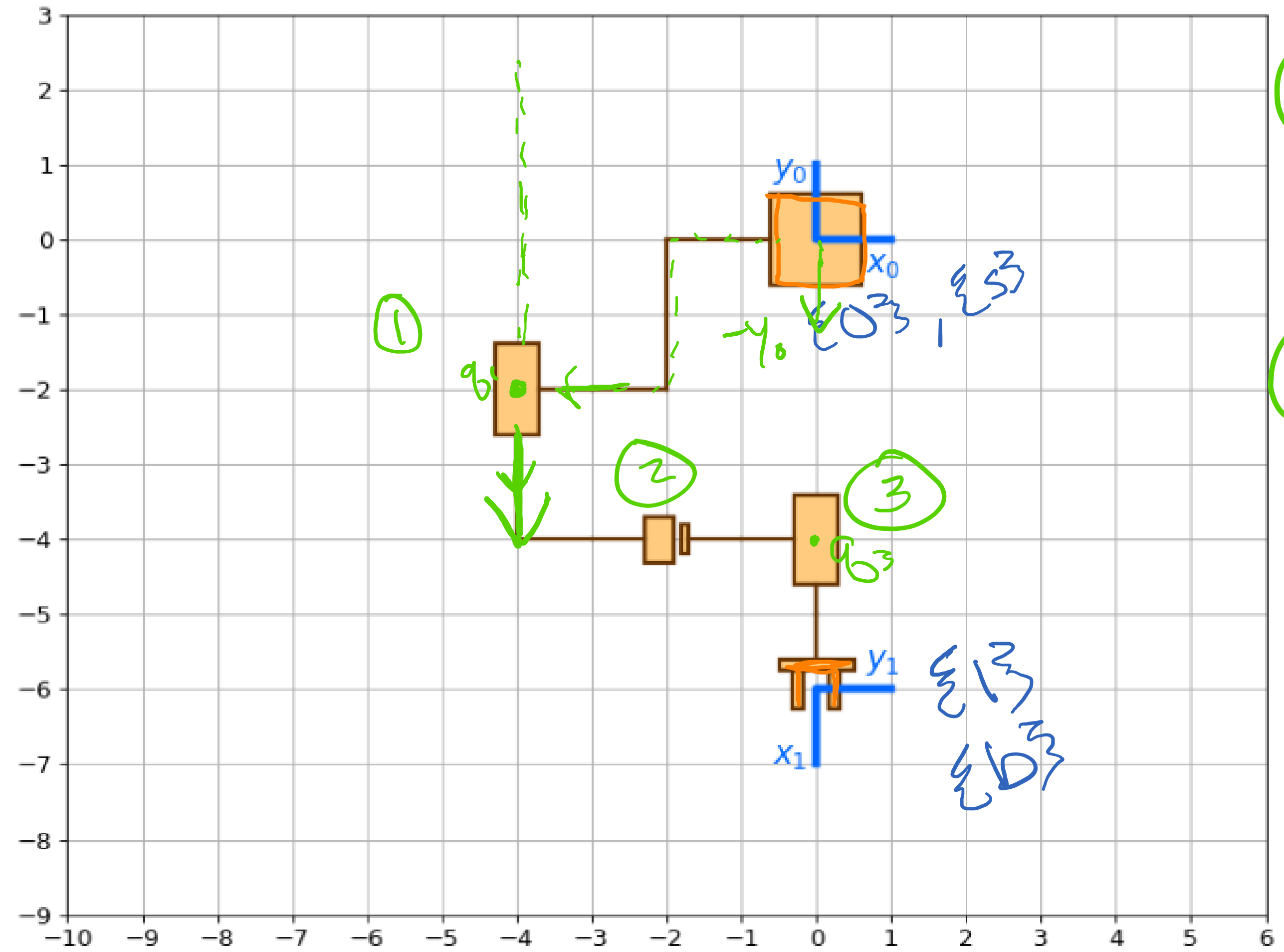
$$e^{[\omega]\theta} = I + [\omega]\theta + [\omega]^2 \frac{\theta^2}{2!} + [\omega]^3 \frac{\theta^3}{3!} + \cdots$$

$$e^{[\omega]\theta} = I + \sin\theta\,[\omega] + (1-\cos\theta)[\omega]^2$$

Example (1)

$$a \times b = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

$$M = \begin{bmatrix} R & P \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



1 $\omega_1 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$, $v_1 = -\omega_1 \times q_1$ $= \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$, $q_1 = \begin{bmatrix} -4 \\ -2 \\ 0 \end{bmatrix}$ | $S_1 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 4 \end{bmatrix}$

2 $\omega_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ | $S_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

3 $\omega_3 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$, $q_3 = \begin{bmatrix} 0 \\ -4 \\ 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$S_3 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$S = \begin{bmatrix} S_1 & S_2 & S_3 \end{bmatrix}$

PoE in the End-Effector Frame

recall some identities:

$$e^{M^{-1}PM} = M^{-1} e^{P} M \rightsquigarrow \underline{e^{P} M = M e^{M^{-1}PM}}$$

$$\begin{aligned} T(\theta) &= e^{[S_1]\theta_1} \cdots e^{[S_n]\theta_n} M \\ &= e^{[S_1]\theta_1} \cdots M e^{M^{-1}[S_n]M\theta_n} \\ &= e^{[S_1]\theta_1} \cdots M e^{M^{-1}[S_{n-1}]M\theta_{n-1}} e^{M^{-1}[S_n]M\theta_n} \\ &= M e^{M^{-1}[S_1]M\theta_1} \cdots e^{M^{-1}[S_n]M\theta_n} \end{aligned}$$

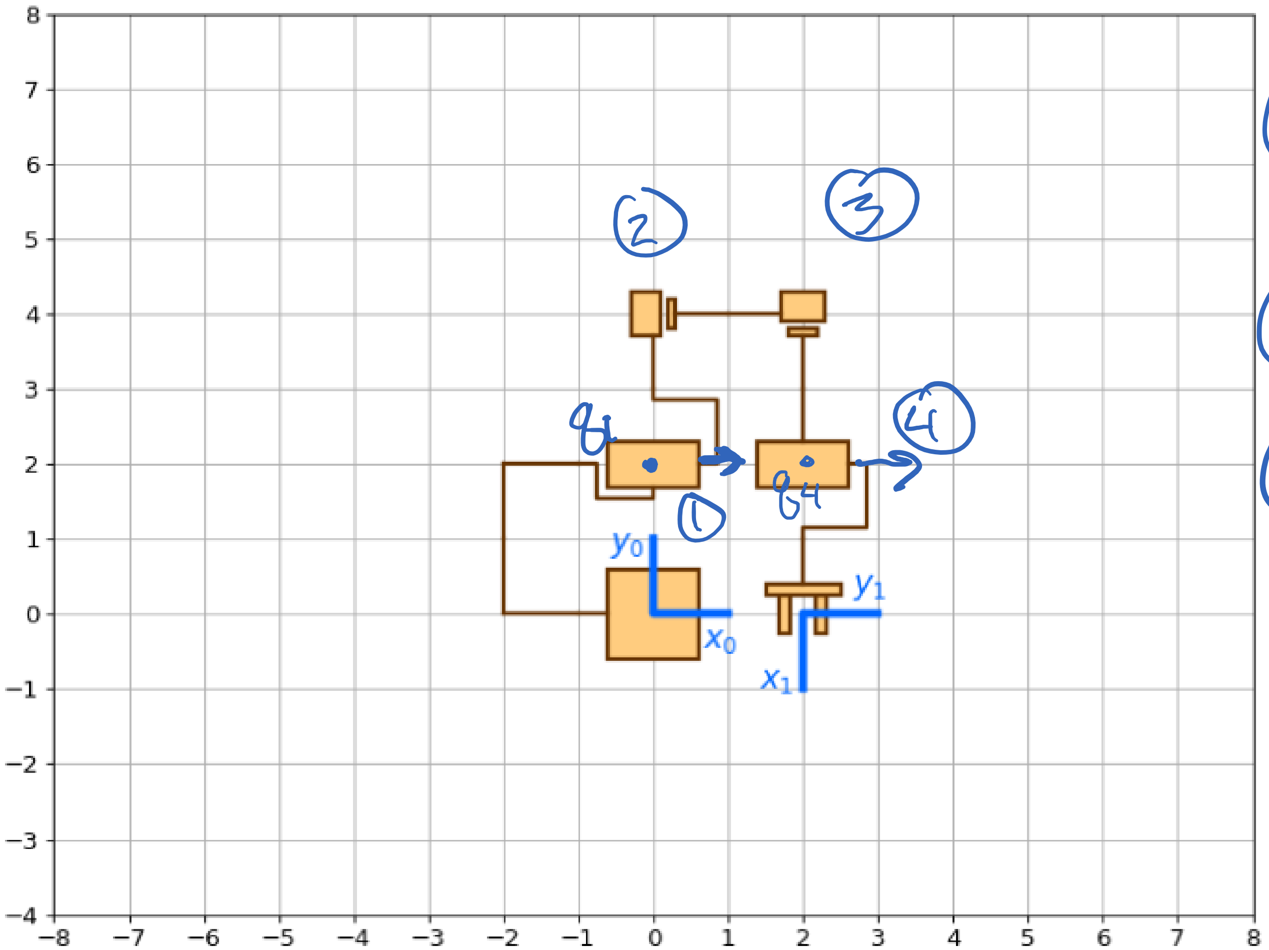
what is $M^{-1}[S_i]M$? $\rightsquigarrow [B_i] \rightarrow$ screw axis in body frame

Body form of POE: $T(\theta) = M e^{[B_1]\theta_1} \cdots e^{[B_n]\theta_n}$

Body Form

Example (2)

$$M = \begin{bmatrix} R & P \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 2 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



① $\omega_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, q_1 = \begin{bmatrix} -2 \\ -2 \\ 0 \end{bmatrix}, v_1 = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$

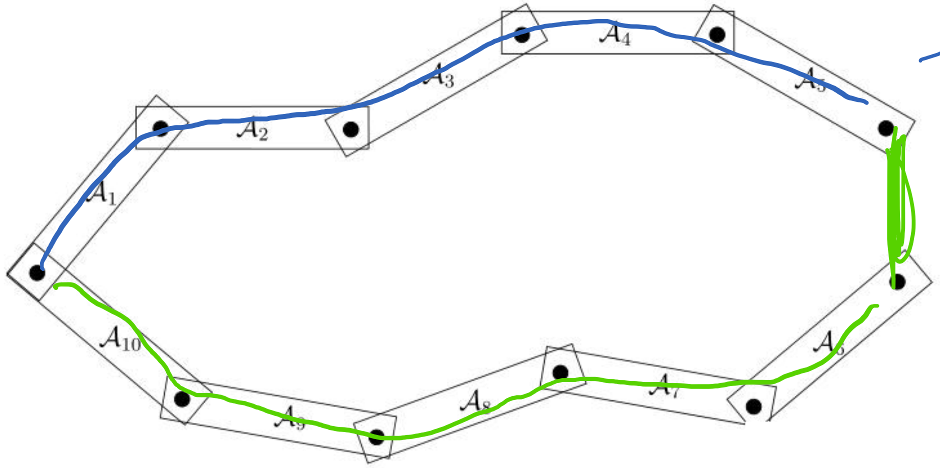
$$B_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ -2 \end{bmatrix}$$

② $\|\omega_2\| = 0, \; v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

③ $\|\omega_3\| = 0, \; v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

④ $\omega_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, q_4 = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}, v_4 = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$

What about closed kinematic chains?

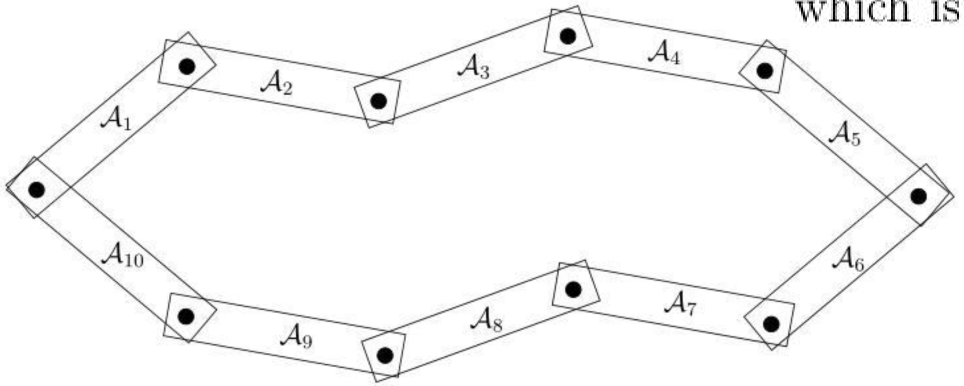


$$T_1(\theta_1)T_2(\theta_2)T_3(\theta_3)T_4(\theta_4)T_5(\theta_5)\begin{pmatrix} a_5 \\ 0 \\ 1 \end{pmatrix},$$

$$T_{10}(\theta_{10})T_9(\theta_9)T_8(\theta_8)T_7(\theta_7)T_6(\theta_6)\begin{pmatrix} a_6 \\ 0 \\ 1 \end{pmatrix},$$

$$T_1(\theta_1)T_2(\theta_2)T_3(\theta_3)T_4(\theta_4)T_5(\theta_5)\begin{pmatrix} a_5 \\ 0 \\ 1 \end{pmatrix} = T_{10}(\theta_{10})T_9(\theta_9)T_8(\theta_8)T_7(\theta_7)T_6(\theta_6)\begin{pmatrix} a_6 \\ 0 \\ 1 \end{pmatrix}, \tag{3.81}$$

which is quite a mess of nonlinear, trigonometric equations that must be solved.



Discussion and Summary

- Our **fancy screw motion matrix exponential** is just another way to write down **homogenous transformations in three dimensions!**
- Our usual homogenous transformation matrices can also be used for forward kinematics (the Denavit-Hartenberg (DH) representation)
  - We define a frame for each link in the frame of the previous link. So to compute the position of the end effector, a frame for each link must be defined in terms of the previous link
- **With screw motions, we have only two reference frames** (base and tool), and then each joint screw motion is defined in the base frame
- And now introduction to **URDF!**