lecture 08
transformations, twists, screws, and wrenches

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Modern Robotics Ch. 3.3-3.4
Admin

• HW4 due on Friday
• Quiz 1 is live! 2/18-2/20 at the CBTF
  • Re-take will be next week
  • Different questions, but same material is covered
Who is Michael Chasles?

- Rodrigues and Chasles took the entrance exam to Polytechnique/Normale at the same time, finishing first and second respectively
  - Rodrigues did not use it and elected to go to La Sorbonne
- The proof that a spatial displacement can be decomposed into a rotation and slide around and along a line is attributed to the astronomer and mathematician Giulio Mozzi (1763),
  - In Italy, the screw axis is traditionally called *asse di Mozzi*
- However, most textbooks refer to a subsequent similar work by Michel Chasles dating 1830
- Several other scholars/contemporaries of M. Chasles obtained the same or similar results around that time, including G. Giorgini, Cauchy, Poinsot, Poisson, and Rodrigues
Homogeneous Transformations: $SE(3)$

The **special Euclidean group** $SE(3)$ is the set of $4 \times 4$ matrices of the form:

$$T = T(R, p) = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \in SE(3)$$

- The inverse of $T$ is $T^{-1} = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix} \in SE(3)$
- If $T_1$ and $T_2 \in SE(3)$, then $T_1 T_2 \in SE(3)$
- $T$ satisfies: $\|Tx - Ty\| = \|x - y\|$ and $$(Tx - Tz)^\top (Ty - Tz) = (x - z)^\top (y - z)$$

Recall homogeneous coordinates: if $x \in \mathbb{R}^3 \rightarrow \begin{bmatrix} x \\ 1 \end{bmatrix}$
Representing a configuration with SE(3)

Each frame can represent a body frame in a multi-link mechanism

As before:

\[ T_{ab} T_{bc} = T_{ac} \]
\[ T_{ab} v_b = v_a \]
Displacing Frames

consider frame $T_{sb}$, a rotation $\text{Rot}(\hat{\omega}, \theta)$, and translation $\text{Trans}(p)$

$\rightarrow$ which is applied first? depends!

$\text{Rot}(\hat{\omega}, \theta) = \begin{bmatrix} R & \theta \\ \hat{o} & 1 \end{bmatrix}$, $\text{Trans}(p) = \begin{bmatrix} 1 & p \\ \hat{o} & 1 \end{bmatrix}$

$T(R, p) = \begin{bmatrix} R & P \\ \hat{o} & 1 \end{bmatrix} = \text{Trans}(p) \cdot \text{Rot}(\hat{\omega}, \theta)$

1. $\hat{\omega}$-axis + $p$ in fixed frame

$T_{s} = T T_{sb} = \text{Trans}(p) \underline{\text{Rot}(\hat{\omega}, \theta)} T_{sb} = \begin{bmatrix} R R_{sb} & R_{psb} + p \\ \hat{o} & 1 \end{bmatrix}$

2. $\hat{\omega}$-axis + $p$ in body frame

$T_{sb}^\prime = T_{sb} T = T_{sb} \underline{\text{Trans}(p)} \underline{\text{Rot}(\hat{\omega}, \theta)} = \begin{bmatrix} R_{sb} R & R_{sb} p + p_{sb} \\ \hat{o} & 1 \end{bmatrix}$
Example Displacement

\[ \hat{\omega} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \]

\[ \theta = 90^\circ \]

\[ \rho = \begin{bmatrix} 0 \\ 0 \\ \frac{2}{3} \end{bmatrix} \]
Mobile arm example

• Robot arm mounted on wheeled platform. Camera fixed to ceiling.

• \{b\} is body frame, \{c\} end-effector frame, \{e\} frame of object, and \{a\} is fixed frame.

• We assume the camera position and orientation in \{a\} is given.

• From camera measurements, you can evaluate the position and orientation of the body and the object in the camera frame.

• Since we designed our robot and have joint-angle estimates, we can obtain the end-effector position and orientation in the body frame.

• To pick up the object, we need the object position and orientation in the frame of our end-effector.
Moving Frames: linear and angular velocity

\[ T_{sb}(t) = T(t) = \begin{bmatrix} R(t) & \dot{p}(t) \\ 0 & 1 \end{bmatrix} \Rightarrow \dot{T} = \begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix} \]

\[ T^{-1} \dot{T} = \begin{bmatrix} R^T & -R^T \dot{p} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} R^T \dot{R} & R^T \dot{p} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \omega_b & v_b \\ 0 & 0 \end{bmatrix} \]

\( w_b + v_b \) give us a 6 parameter rep of body twist

\( \rightarrow \) spatial velocity in \( \mathbb{E}^3 \)

\[ \gamma_b = \begin{bmatrix} w_b \\ v_b \end{bmatrix} \]

bracket notation: \[ [\gamma_b] = \begin{bmatrix} [w_b] & v_b \\ 0 & 0 \end{bmatrix} e_s e^s(3) \]

note: 1 different bracket, 2 \( [v_b] \) is skew-symm
Linear and angular velocity in ref. frame

\[ \dot{T} T^{-1} = \begin{bmatrix} \hat{R} & \dot{p} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R^T & -R^T \dot{p} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \hat{R} R^T & \dot{p} - \hat{R} R^T \dot{p} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} [\omega_s] & \dot{v}_s \\ 0 & 0 \end{bmatrix} \]

**Spatial twist:** Spatial vel in $\mathbb{SE}(3)$ frame

\[ \mathbf{v}_s = \begin{bmatrix} \omega_s \\ \dot{v}_s \end{bmatrix}, \quad [\mathbf{v}_s] = \begin{bmatrix} \omega_s & \dot{v}_s \\ 0 & 0 \end{bmatrix} \in \mathbb{SE}(3) \]

\[ \mathbf{v}_0 = \dot{p} - \hat{R} R^T \dot{p} = \dot{p} - [\omega_s] p = \dot{p} - \omega_s \times p \]
Physical interpretation of $v_s$

• Assume that a very large rigid body is attached to the frame \{b\}, and is large enough to contain the origin of \{s\}

• What is the velocity of the point of this body as the origin of \{s\}?

• There are two components:
  • $\dot{p}$: the motion of the body
  • $\omega_s \times (-p)$: the rotation of the body
Twists

\[ \begin{align*}
[Y_b] &= T^{-1} [Y_b^*] T \quad \text{and} \\
[Y_s] &= T [V_b] T^{-1}
\end{align*} \]

\[ [Y_s] = \begin{bmatrix}
R [w_b] R^T & -R [w_b] R^T p + R v_b \\
0 & 0
\end{bmatrix} \]

A trick: \[ [w] p = w \times p = -p \times w = -[p] w \]

\[ V_s = [w_s] = \begin{bmatrix}
[p] R & 0 \\
0 & R
\end{bmatrix} [w_b] \]

\footnote{Properties: 1. \( \text{Ad}_{T_1}(\text{Ad}_{T_2}(Y)) = \text{Ad}_{T_1 T_2}(Y) \)}

\[ V_1 = \text{Ad}_T(Y) \iff [V_1^*] = T [V_1] T^{-1} \]

\[ \text{Ad}_T(Y) \mapsto \text{adjoint map of } T \]

Recall: \( T = T_{sb} \)
The twist corresponding to the instantaneous motion of the chassis of a three-wheeled vehicle can be visualized as an angular velocity $\omega$ about the point $r$. 
Screw Motions
2D Screw Motions

• Any rigid motion in the plane can be represented by a rotation around a well-chosen center.

• We can encode it with \((\beta, s_x, s_y)\), where \((s_x, s_y)\) is the position of the center of the rotation, and \(\beta\) the angle.

• Recall that the angular velocity \(\omega\) can be viewed as \(\hat{\omega} \dot{\theta}\), where \(\hat{\omega}\) is the unit rotation axis and \(\dot{\theta}\) is the rate of rotation.

• A twist \(\nu\) can be interpreted in terms of a screw axis \(S\) and a velocity \(\dot{\theta}\) about that axis.
Screw motions in 3D

**Chasles-Mozzi Theorem:**
Any displacement in 3D can be represented by a rotation and translation about the same axis, referred to as a screw motion

\[ q \in \mathbb{R}^3 \] is any point along the axis
\[ \hat{s} \] is a unit vector in the direction of the axis
\( h \) is the **screw pitch**, which is the ratio between the linear and angular speed along axis

We write this as \( S = \{ q, \hat{s}, h \} \)
Screw motions and Twists

\[
y = \begin{bmatrix} \omega \\ \nu \end{bmatrix} = \begin{bmatrix} \Sigma \hat{\Theta} \\ h \hat{\Theta} - \hat{\Theta} \times q \end{bmatrix}
\]

- Trans along screw axis
- Linear motion at origin induced by rotation of screw axis

\[
\hat{s} = \frac{w}{||w||} = \hat{\omega}, \quad \hat{\Theta} = \frac{w}{||w||}, \quad h = \frac{\hat{\omega}^T \nu}{\hat{\Theta}} = \frac{\nu}{||w||^2}
\]

Assuming \(w \neq 0\), find \(q\) s.t.:
\[
v = h \hat{s} \hat{\Theta} - \hat{s} \hat{\Theta} \times q \quad \rightarrow \quad q = \frac{\hat{s} \times v}{\hat{\Theta}}
\]
Screw axis

Given a reference frame, a screw axis $S$ is defined:

$$S = \begin{bmatrix} w \\ v \end{bmatrix} e^{\mathbf{TR}_a}$$

where either:

1. $||w|| = 1$
   
   $$v = -w \times q + hw$$

   $h = 0$ gives pure rotation (this is useful for revolute joints)

2. $w = 0$ + $||v|| = 1$
   
   $L \rightarrow$ translating along $v$
   
   $$[S] = \begin{bmatrix} [w] & v \\ 0 & 0 \end{bmatrix} e^{\mathbf{TR}_a} \in SE(3)$$

   $$S_a = [\text{Ad}_{T_{a0}}] S_b$$

note that these are overloaded variables for screws, either $||w||$ or $||v||$ must be 1
$\Rightarrow$ $y$ uses $w + v$ too, but these are unconstrained!
Exponential Coordinates for Rigid Body Motions

Recall for rotations: $R = e^{[\omega]\theta}$

What are exp coordinates for $T$?

$S \in \mathbb{R}^6$, $S$ is screw axis and $\theta$ is distance about $S$ to take frame from $I$ to $T$

$e^{[S] \theta} = I + [S] \theta + [S]^2 \theta^2/2! + \cdots$

$e^{[\omega] \theta} = G(\theta) = I \theta + [\omega] \theta^2/2! + \cdots$

$G(\theta) = I \theta + (\theta^2/2! - \cdots) [\omega] + (\theta^3/3! - \cdots) [\omega]^2$

$= I \theta + (1 - \cos \theta) [\omega] + (\theta - \sin \theta) [\omega]^2$
Logarithm of rigid body motions

Given $T(R,p) \in SE(3)$, find $S = [\omega, v] + \Theta$ s.t.

$$e^{[S]\Theta} = \begin{bmatrix} R & p \end{bmatrix} \rightarrow \begin{bmatrix} S \end{bmatrix}\Theta = \begin{bmatrix} [\omega]\Theta & v\Theta \end{bmatrix}$$

logarithm of $T$

1. if $R=I$, then $\omega=0$ and $v = \frac{p}{||p||}$ and $\Theta = ||p||$

2. find $\omega, \Theta$ using matrix log of $R$ (lecture 7)
   
   Then, $v = G^{-1}(\Theta)p$
   
   $\mapsto \frac{1}{\Theta}I - \frac{1}{2}[\omega] + (\frac{1}{\Theta} - \frac{1}{2} \cot \Theta \Theta)[\omega]^2$

this allows us to go back and forth between exp coordinates and transformations!
Summary

• Formally defined **homogeneous transformations**
• Used transformations to define spatial velocities as the **body and spatial twist**
• Introduced **screw motions** and the screw interpretation of a twist
• The **screw axis S** was defined and used to define the **exponential coordinates of homogeneous transformations**
Wrenches

• Let \( \{a\} \) be a ref. frame and \( r_a \) a point in a rigid body

• Suppose we have a force acting on the body at point \( r_a \), represented by vector \( f_a \in \mathbb{R}^3 \)

• This force creates a torque or moment:
  \[
  m_a = r_a \times f_a \in \mathbb{R}^3
  \]

• We introduce a spatial force or wrench: \( \mathcal{F}_a := \begin{bmatrix} m_a \\ f_a \end{bmatrix} \)
Wrenches in two frames

Suppose I want to express a wrench in frames A and B. \( \mathbf{v}_a, \mathbf{v}_b \) are the motions induced by the wrench. This is independent of the frame.

Recall: \( \mathbf{v}_b = \text{Ad}_{\mathbf{T}_{ab}}(\mathbf{v}_a) \)

Also recall: dot product of velocity and force \( \mathbf{v}_b \mathbf{F}_b = \mathbf{v}_a \mathbf{F}_a = (\text{Ad}_{\mathbf{T}_{ab}})^T \mathbf{F}_a = \mathbf{v}_b [\text{Ad}_{\mathbf{T}_{ab}}]^T \mathbf{F}_a \)

\[ \Rightarrow \mathbf{F}_b = [\text{Ad}_{\mathbf{T}_{ab}}]^T \mathbf{F}_a \]

Similarly: \( \mathbf{F}_a = [\text{Ad}_{\mathbf{T}_{ba}}]^T \mathbf{F}_b \)