Multi-objective control for multi-agent systems using Lyapunov-like barrier functions

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Abstract— This paper addresses the problem of multi-agent coordination and control under multiple objectives, and presents a set-theoretic formulation which is amenable to Lyapunov-based analysis and control design. A novel class of Lyapunov-like barrier functions is introduced and used to encode multiple, non-trivial control objectives, such as collision avoidance, proximity maintenance and convergence to desired destinations. The construction is based on the concept of recentered barrier functions and on approximation functions. A single Lyapunov-like function encodes the constrained set of each agent, yielding simple, closed-form control solutions. The proposed construction allows also for distributed control design based on information locally available to each agent. The scenario considered here involves nonholonomic vehicles, while simulation results demonstrate the efficacy of the approach.

I. INTRODUCTION

Research in multi-agent robotic systems has seen increased interest during the past decade, motivated in part by applications such as surveillance, patrolling, exploration and coverage. Coordination and control in such casesl is naturally dictated by the available patterns on sensing and information sharing, as well as by physical/environmental constraints and inherent limitations (e.g. motion constraints, obstacles, unmodeled disturbances, input saturations etc). It is out of the scope of this paper to provide an overview of the existing methodologies; the reader is referred, for instance, to [1], [2].

The main objectives in a multi-robot network are pertinent to inter-agent collision avoidance, convergence to spatial destinations/regions or tracking of reference signals/trajectories, maintenance of information exchange among agents and avoidance of physical obstacles. Such objectives are encountered in flocking [3]–[6], consensus, rendezvous and/or formation control [7]–[11]. Collision avoidance is an un-negotiable requirement in such problems, and is often addressed with potential function methods and Lyapunov-based analysis. It is worth mentioning that these contributions give emphasis in some of, i.e., not in all at once, the aforementioned control objectives. Furthermore, lately there has been significant interest in the deployment of robotic

networks for exploration, surveillance and patrolling of inaccessible, dangerous or even hostile environments, such as oil drilling platforms, nuclear reactors, border changes etc [12]–[15], offering a plethora of formulations and solutions, from combinatorial motion planning to optimization-based and Lyapunov-based methods.

This paper is motivated in part by surveillance applications which bring in the need for multi-agent coordination and control algorithms under multiple objectives, and introduces a set-theoretic formulation [16] which is amenable to Lyapunov-like analysis and control design. More specifically, a novel class of Lyapunov-like barrier functions is introduced and used to encode multiple, non-trivial control objectives, such as collision avoidance, proximity maintenance and convergence to desired destinations. The construction is based on the concept of recentered barrier functions [17] and on the approximation functions introduced in earlier work of the authors' [18]. One of the merits of the approach is that a single Lyapunov-like function is used to encode the constrained set of each agent, yielding thus simple, gradient-based, closed-form control solutions. The proposed construction allows for distributed control design based on information locally available to each agent, which may be dictated by limited sensing capabilities or given communication topologies. The scenario considered here involves nonholonomic vehicles with unicycle kinematics, while the derived control laws for each agent require information exchange among neighbor agents regarding on their positions only, i.e., the agents do not need to exchange their full state (pose and velocities), as often assumed in similar control designs.

The paper is organized as follows: Section II gives the mathematical modeling and problem formulation and Section III presents the objective encoding via the novel Lyapunov-like barrier functions. The motion coordination and control design is addressed in Section IV while simulation results demonstrate the efficacy of our approach in Section V. Conclusions and ideas on current and future research are summarized in Section VI.

II. MODELING AND PROBLEM STATEMENT

Consider a network of N mobile agents with unicycle kinematics, which is deployed in a known workspace (environment) $\mathcal{W}$ with static obstacles. Each agent $i \in \{1, \ldots, N\}$ is modeled as a circular disk of radius r_0, and its motion with respect to (w.r.t.) a global cartesian coordinate frame $\mathcal{G}$ is described by:

$$\dot{\boldsymbol{q}}_i = \begin{bmatrix} \cos\theta_i & \sin\theta_i & 0 \end{bmatrix}^T u_i + \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T \omega_i, \quad (1)$$

where $\boldsymbol{q}_i = \begin{bmatrix} x_i & y_i & \theta_i \end{bmatrix}^T \in \mathcal{Q}_i$ is the configuration vector of agent i, comprising the position $\boldsymbol{r}_i = \begin{bmatrix} x_i & y_i \end{bmatrix}^T \in \mathcal{R}_i$ and the orientation $\theta_i \in \mathcal{S}$ of agent i, $\mathcal{Q}_i = \mathcal{R}_i \times \mathcal{S}$ is the configuration space of agent i, and $\boldsymbol{\nu}_i$ is the vector of control inputs, comprising the linear velocity u_i and the angular velocity ω_i, expressed in the body-fixed frame $\mathcal{B}_i$.

A. Network structure

We assume that each agent i has access to its own configuration $\boldsymbol{q}_i \in \mathcal{Q}_i$ via onboard sensors, and that it can reliably transmit this information to any agent j which lies within a maximum distance $2R_0$ via wireless communication links. In other words, a pair of agents (i, j) remains connected as long as the distance d_{ij} remains smaller than $2R_0$.

We assign agent $i = 1$ to be the leader of the network. This is actually realistic and relevant since typically, the leader of a robotic group may be responsible for computing high-level collision-free motion plans either for the whole group, or for itself only, and for coordinating the motion of the remaining agents (followers) according to the updated motion plans and/or other objectives, etc.

B. Control objectives

The agents need to accomplish a set of control objectives while operating in the environment $\mathcal{W}$. The leader agent can be seen as the agent of highest priority, which should:
(L1) converge to a goal destination or track a reference trajectory,
(L2) remain in the proximity of the followers in order to effectively broadcast updated motion plans,
(L3) while *not deviating from its nominal motion plan or trajectory*.
Note that the leader is not assigned with the objective of avoiding collisions with the remaining $N - 1$ agents. Consequently, the followers should move while ensuring:
(F1) safety, which is realized as the guaranteed avoidance of collisions among agents,
(F2) connectivity maintenance, which is realized as keeping pairwise upper bounded distances, so that information exchange remains reliable, and
(F3) convergence to, or tracking, goal destinations, so that they ultimately surveil the area.

III. LYAPUNOV-LIKE BARRIER FUNCTIONS

In constrained optimization, a barrier function is a continuous function whose value on a point increases to infinity as the point approaches the boundary of the feasible region; therefore, a barrier function is used as a penalizing term for violations of constraints. The concept of *recentered barrier functions* was introduced in [17] in order to not only regulate the solution to lie in the interior of the constrained set, but also to ensure that, if the system converges, then it converges to a desired point.

A. Collision avoidance

Agent $i \in \{2, \ldots, N\}$ realizes agent $j \in \{1, \ldots, N\}$, $j \neq i$, as a physical obstacle. Therefore, agent i avoids collision with agent j as long as the distance $d_{ij} = \|\boldsymbol{r}_i - \boldsymbol{r}_j\|$ remains

greater or equal than a minimum separation distance $d \geq 2r_0$, i.e., as long as:

$$c_{ij} = (x_i - x_j)^2 + (y_i - y_j)^2 - d^2 \geq 0. \qquad (2)$$

The inequality (2) is essentially a nonlinear inequality constraint which should never be violated. In the sequel, for all $i \in \{2, \ldots, N\}$ and all $j \in \{1, \ldots, N\}$, with $j \neq i$, we refer to the resulting $(N-1) \times (N-2)$ constraints as to *collision avoidance* constraints, while the constrained set encoding the collision-free space of agent i w.r.t. agent j is denoted with $\mathcal{K}_{ij} = \{\boldsymbol{r}_i \in \mathcal{R}_i, \boldsymbol{r}_j \in \mathcal{R}_j \mid c_{ij}(\cdot) \geq 0\}$.

In order to encode the collision avoidance constraints via Lyapunov-like functions, we define the logarithmic barrier function[1] $b_{ij}(\cdot) : \mathcal{R}_i \times \mathcal{R}_j \to \mathbb{R}$ for the constraint $c_{ij}(\cdot)$ as:

$$b_{ij}(\cdot) = -\ln\left(c_{ij}(\cdot)\right),$$

which tends to $+\infty$ as $c_{ij}(\cdot) \to 0$. The *gradient recentered barrier function* for the constraint $c_{ij}(\cdot)$ is given as [17]:

$$r_{ij}(\cdot) = b_{ij}(\boldsymbol{r}_i, \boldsymbol{r}_j) - b_{ij}(\boldsymbol{r}_{id}, \boldsymbol{r}_j) - \nabla b_{ij}(\boldsymbol{r}_{id}, \boldsymbol{r}_j)^T \delta \boldsymbol{r}_i, \quad (3)$$

where: $\delta \boldsymbol{r}_i \triangleq \boldsymbol{r}_i - \boldsymbol{r}_{id}$, $\boldsymbol{r}_{id} = \begin{bmatrix} x_{id} & y_{id} \end{bmatrix}^T$ is the goal position of agent i, $\nabla b_{ij} = \begin{bmatrix} \frac{\partial b_{ij}}{\partial x_i} & \frac{\partial b_{ij}}{\partial y_i} \end{bmatrix}^T$ is the gradient (column) vector of the function $b_{ij}(\cdot)$, and $\nabla b_{ij}(\boldsymbol{r}_{id}, \boldsymbol{r}_j)^T$ is the transpose of the gradient vector (i.e., is a row vector, for the dimensions to match) *evaluated at the goal position* $\boldsymbol{r}_{id}$. By construction, the recentered barrier function (3):
1) is non-zero everywhere within the constrained set $\mathcal{K}_{ij}$ *except for* the goal position $\boldsymbol{r}_{id}$ of agent i, and
2) tends to $+\infty$ at the boundary of the constrained set $\mathcal{K}_{ij}$, i.e., when the distance $d_{ij} \to d$.

Motivated by these characteristics, the main idea here is to employ the recentered barrier function (3) in order to encode *both* collision avoidance of agent i w.r.t. agent j *and* convergence of agent i to a goal position $\boldsymbol{r}_{id}$. In order to ensure that we have an everywhere nonnegative function encoding these objectives, so that it can be used in Lyapunov-like control design and analysis according to [18], we define:

$$V_{ij}(\cdot) = (r_{ij}(\cdot))^2, \qquad (4)$$

which now is a positive definite function $V_{ij}(\boldsymbol{r}_i, \boldsymbol{r}_j) : \mathcal{R}_i \times \mathcal{R}_j \to \mathbb{R}^+$ for agent i; more specifically, $V_{ij}(\boldsymbol{r}_i, \boldsymbol{r}_j)$ is zero at the goal position $\boldsymbol{r}_{id}$, and tends to infinity as $r_{ij}(\cdot) \to +\infty$, i.e., on the boundary of the constrained set $\mathcal{K}_{ij}$.

B. Connectivity maintenance

Pairwise information exchange among agents i, j is reliable as long as the inter-agent distance d_{ij} remains less or equal than a maximum distance $2R_0$. This requirement is equivalent with ensuring that all N agents remain within a circular region O of center $\boldsymbol{r}_0 = \begin{bmatrix} x_0 & y_0 \end{bmatrix}^T$ and radius R_0, i.e., as long as the distance $d_{i0} = \|\boldsymbol{r}_i - \boldsymbol{r}_0\|$ remains always less or equal than $R_0 - r_0$, which reads:

$$c_{i0} = (R_0 - r_0)^2 - (x_i - x_0)^2 - (y_i - y_0)^2 \geq 0. \qquad (5)$$

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[1]The choice of the logarithmic barrier function is not restrictive; one may use other types of barriers, e.g. the inverse barrier function $b_{ij} = \frac{1}{c_{ij}}$.

In the sequel, for all $i \in \{1, \dots, N\}$ we refer to the N constraints (5) as to *proximity* constraints, while the constrained set encoding the circular region O, or the connectivity region, for each agent i is denoted as $\mathcal{K}_{i0} = \{\boldsymbol{r}_i \in \mathcal{R}_i \mid c_{i0}(\cdot) \geq 0\}$. Connectivity for the robotic network is then maintained as long as the constraints (5) hold for all $i \in \{1, \dots, N\}$.

The proximity constraints are encoded following the idea described earlier. For each agent i we define the logarithmic barrier function $b_{i0}(\cdot) : \mathcal{R}_i \to \mathbb{R}$ of the constraint $c_{i0}(\cdot)$ as:

$$b_{i0}(\cdot) = -\ln\left(c_{i0}(\cdot)\right),$$

and the corresponding recentered barrier function as:

$$r_{i0}(\cdot) = b_{i0}(\boldsymbol{r}_i, \boldsymbol{r}_0) - b_{i0}(\boldsymbol{r}_{id}, \boldsymbol{r}_0) - \nabla b_{i0}(\boldsymbol{r}_{id}, \boldsymbol{r}_0)^T \delta \boldsymbol{r}_i, \quad (6)$$

where: $\nabla b_{i0}(\boldsymbol{r}_{id}, \boldsymbol{r}_0)^T$ is the transpose of the gradient vector $\nabla b_{i0} = \left[\frac{\partial b_{i0}}{\partial x_i} \quad \frac{\partial b_{i0}}{\partial y_i}\right]^T$ of the inverse barrier function $b_{i0}(\cdot)$, evaluated at $\boldsymbol{r}_{id}$. The recentered barrier function $r_{i0}(\cdot)$ vanishes at the goal position $\boldsymbol{r}_{id}$ of agent i and tends to $+\infty$ as $c_{i0}(\cdot) \to 0$, i.e., as the agent i approaches the boundary of the constrained set $\mathcal{K}_{i0}$. For encoding the proximity objective for agent i by a nonnegative function we define:

$$V_{i0}(\cdot) = \left(r_{i0}(\cdot)\right)^2, \quad (7)$$

which yields the positive definite function $V_{i0}(\boldsymbol{r}_i, \boldsymbol{r}_{id}, \boldsymbol{r}_0)$; this function is zero at the desired position $\boldsymbol{r}_{id}$ of agent i, and tends to infinity as $r_{i0}(\cdot) \to +\infty$, i.e., on the boundary of the connectivity region O.

C. Bounded barrier functions for collision avoidance, proximity and convergence objectives

The analytical construction and properties of the squared recentered barrier functions (4), (7) allow for handling the collision avoidance, proximity and convergence objectives via a single Lyapunov-like function V_i for each agent i. Following previous work of ours' [18], we encode the accomplishment of all objectives for agent i by an approximation of the maximum function (which also is a δ-norm when δ takes integer values), of the form:

$$v_i = \left((V_{i0})^\delta + \sum_{j=1, j \neq i}^{N} (V_{ij})^\delta\right)^{\frac{1}{\delta}}, \quad (8)$$

where $\delta \in [1, +\infty)$. The function v_i vanishes when $V_{ij} = V_{i0} = 0$, i.e., at the goal destination $\boldsymbol{r}_{id}$, and tends to $+\infty$ as at least one of the terms V_{ij}, V_{i0} tends to $+\infty$, i.e., as the position $\boldsymbol{r}_i$ of agent i approaches the boundary of the constrained set $\mathcal{K}_i = \bigcup_{j=0, j \neq i}^{N} \mathcal{K}_{ij}$.

Remark 1: The proposed encoding may easily incorporate collision avoidance of agent i w.r.t. all or a subset of agents $j \neq i$. For instance, one may take into account the "neighbor agents" j only when defining (8), according to local sensing limitations or given communication topologies.

Finally, for ensuring that all objectives are encoded by a single function which *uniformly* attains its maximum value

on the boundary of the constrained set $\mathcal{K}_i$,[2] we define:

$$V_i = \frac{v_i}{1 + v_i} = \frac{\left(\sum_{j=0, j \neq i}^{N} (V_{ij})^\delta\right)^{\frac{1}{\delta}}}{1 + \left(\sum_{j=0, j \neq i}^{N} (V_{ij})^\delta\right)^{\frac{1}{\delta}}}, \quad (9)$$

which is zero for $v_i = 0$, i.e., at the goal position $\boldsymbol{r}_{id}$ of agent i, and equal to 1 as $v_i \to +\infty$, i.e., on the boundary of the constrained set $\mathcal{K}_i$.

IV. MOTION COORDINATION

All agents initiate in the region O, so that reliable wireless communication links can be established. The leader $i = 1$ is responsible for guiding the followers $j \neq i$ through the environment by communicating a goal position $\boldsymbol{r}_{jd}$ to each agent j. We assume for now that the goal positions $\boldsymbol{r}_{jd}$, $j \in \{1, \dots, N\}$, as well as the center $\boldsymbol{r}_0$ of the region O are static, which yields: $\frac{d}{dt}\boldsymbol{r}_{jd} = \boldsymbol{0}$ and $\frac{d}{dt}\boldsymbol{r}_0 = \boldsymbol{0}$, respectively, and that there are no physical obstacles in the region O.

A. Control laws for agent $j = 1$

Proposition 1: The leader agent converges globally to its desired destination $\boldsymbol{r}_{1d}$ while remaining in the region of connectivity O under the control law:

$$u_1 = k_1 \tanh(\|\boldsymbol{r}_1 - \boldsymbol{r}_{1d}\|), \quad (10a)$$
$$\omega_1 = -\lambda_1(\theta_1 - \phi_1) + \dot{\phi}_1, \quad (10b)$$

where $\phi_1 \triangleq \arctan\left(-\frac{\partial V_1}{\partial y_1}, -\frac{\partial V_1}{\partial x_1}\right)$, V_1 is the Lyapunov-like function (9) for $j = 0$, $\delta \geq 1$ and k_1, $\lambda_1 > 0$. The proof is straightforward and omitted here in the interest of space.

B. Control laws for agents $j \in \{2, \dots, N\}$

In order to design control strategies for the follower agents $j \neq 1$ one may employ the function V_j given by (9) as a suitable candidate Lyapunov-like function for each agent j.

Proposition 2: Each agent $j \in \{2, \dots, N\}$ converges almost globally to its desired destination $\boldsymbol{r}_{jd}$, while avoiding collisions w.r.t. agents $k \neq j$ and while remaining in the region of connectivity O, under the control law:

$$u_j = k_j \tanh(\|\boldsymbol{r}_j - \boldsymbol{r}_{jd}\|), \quad (11a)$$
$$\omega_j = -\lambda_j(\theta_j - \phi_j) + \dot{\phi}_j, \quad (11b)$$

where $\phi_j \triangleq \arctan\left(-\frac{\partial V_j}{\partial y_j}, -\frac{\partial V_j}{\partial x_j}\right)$, $\delta \geq 1$ and k_j, $\lambda_j > 0$.

Proof: Let us consider the candidate Lyapunov-like function (9) for agent j, written as:

$$V_j(\boldsymbol{r}_j, \boldsymbol{r}_{jd}, \boldsymbol{r}_k) = \frac{\left(\sum_{k \neq j} V_{jk}{}^\delta\right)^{\frac{1}{\delta}}}{1 + \left(\sum_{k \neq j} V_{jk}{}^\delta\right)^{\frac{1}{\delta}}}, \quad (12)$$

where $k \in \{0, 1, \dots, N\}$, $k \neq j$. The function V_j encodes proximity (or connectivity maintenance) for $k = 0$, collision avoidance w.r.t. the leader for $k = 1$, and collision avoidance w.r.t. the remaining followers for $k \neq j$. Its time derivative along the trajectories of agent j as:

$$\dot{V}_j = \boldsymbol{\zeta}_j^T \begin{bmatrix} \cos\theta_j \\ \sin\theta_j \end{bmatrix} u_j + \sum_{k=1, k \neq j}^{N} \left(\boldsymbol{\zeta}_{jk}^T \begin{bmatrix} \cos\theta_k \\ \sin\theta_k \end{bmatrix} u_k\right), \quad (13)$$

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[2]The reason for requiring this property is justified in Lemma 2.

where $\boldsymbol{\zeta}_j \triangleq \left[\frac{\partial V_j}{\partial x_j} \quad \frac{\partial V_j}{\partial y_j}\right]^T$, $\boldsymbol{\zeta}_{jk} \triangleq \left[\frac{\partial V_j}{\partial x_k} \quad \frac{\partial V_j}{\partial y_k}\right]^T$. Assume that the orientation trajectories $\theta_j(t)$ of agent j are controlled at a much faster time scale via the control law (11b) compared to the position trajectories $\boldsymbol{r}_j(t)$. The system dynamics of agent j can then be decomposed into two subsystems, with the dynamics along the position trajectories $\boldsymbol{r}_j(t)$ serving as the reduced (slow) subsystem, and the orientation dynamics serving as the boundary-layer (fast) subsystem. In order to employ a singular perturbations argument, consider the sufficiently small parameter $\varepsilon_j \triangleq \frac{1}{\lambda_j}$, and rewrite the closed-loop boundary-layer (fast) dynamics as:

$$\varepsilon_j \, \dot{\theta}_j = -(\theta_j - \phi_j) + \varepsilon_j \dot{\phi}_j. \tag{14}$$

Lemma 1: The orientation θ_j of agent j is globally exponentially stable to $\phi_j \triangleq \arctan\left(-\frac{\partial V_j}{\partial y_j}, -\frac{\partial V_j}{\partial x_j}\right)$.

Proof: The roots of the boundary-layer subsystem are given for $\varepsilon_j = 0$; out of (14) one has that the roots of the fast subsystem lie on the manifold $\theta_j = \phi_j$. Denote $\eta_j = \theta_j - \phi_j$ and take the dynamics:

$$\frac{d\eta_j}{dt} = \frac{d}{dt}(\theta_j - \phi_j) \overset{(14)}{=} -\frac{1}{\varepsilon_j}(\theta_j - \phi_j) = -\frac{1}{\varepsilon_j}\eta_j \Rightarrow$$
$$\varepsilon_j \frac{d\eta_j}{dt} = -\eta_j, \text{ which further reads: } \frac{d\eta_j}{d\tau} = -\eta_j,$$

where $\frac{d\tau}{dt} \triangleq \frac{1}{\varepsilon_j}$. The origin $\eta_j = 0$ of the boundary-layer subsystem is thus globally exponentially stable, which implies that θ_j is globally exponentially stable to ϕ_j. ∎

Lemma 2: The control law (11) renders the constrained set $\mathcal{K}_j$ a positively invariant set for the dynamics (1) of agent j. This in turn implies that collision avoidance and proximity for agent j are guaranteed.

Proof: Since the Lyapunov-like function V_j has compact level sets by construction, $\mathcal{K}_j$ is a compact (i.e., closed and bounded) set. Furthermore, on the boundary of the set $\mathcal{K}_j$ the negated gradient vector field $-\boldsymbol{\zeta}_j$ points *by construction* into the interior of $\mathcal{K}_j$; this is ensured since V_j is uniformly maximal on the boundary of $\mathcal{K}_j$. Thus, forcing the orientation θ_j of agent j to globally exponentially track the orientation $\phi_j = \arctan(-\frac{\partial V_j}{\partial y_j}, -\frac{\partial V_j}{\partial x_j})$ via the angular velocity control law (11b) essentially has the effect of forcing agent's j system vector field to point into the interior of the safe (i.e., collision-free and connectivity preserving) compact level sets of V_j. At the same time, the control law (11a) renders an always positive linear velocity u_j, which ensures that agent j always moves in the same direction with the safe gradient vector field dictated by the Lyapunov-like function V_j. Consequently, a straightforward application of Nagumo's theorem [16] implies that, for suitably selected control gains k_j (see Remark 3), the position trajectories $\boldsymbol{r}_j(t)$ starting in $\mathcal{K}_j$ never escape $\mathcal{K}_j$, which in turn implies that the objectives of collision avoidance and proximity for agent j are always accomplished. ∎

For studying the (almost global) convergence to the desired destination $\boldsymbol{r}_{jd}$, let us go back to the derivative of the Lyapunov-like function V_j for the reduced (slow) subsystem,

evaluated at the equilibria $\theta_j = \phi_j$, $\theta_k = \phi_k$ of the boundary-layer (fast) subsystems:

$$\dot{V}_j = \boldsymbol{\zeta}_j^T \begin{bmatrix} \cos\phi_j \\ \sin\phi_j \end{bmatrix} u_j + \sum_{k=1, k\neq j}^{N} \left(\boldsymbol{\zeta}_{jk}^T \begin{bmatrix} \cos\phi_k \\ \sin\phi_k \end{bmatrix} u_k \right). \tag{15}$$

Note that ϕ_j and ϕ_k are by definition the orientations of the vectors $-\boldsymbol{\zeta}_j$ and $-\boldsymbol{\zeta}_k$ respectively. Thus, one can write: $-\frac{\partial V_j}{\partial x_j} = \|\boldsymbol{\zeta}_j\| \cos\phi_j$, $-\frac{\partial V_j}{\partial y_j} = \|\boldsymbol{\zeta}_j\| \sin\phi_j$, where $\|\cdot\|$ stands for the standard Euclidean norm; this holds accordingly for subscript k. Substituting into (15) yields:

$$\dot{V}_j = -\|\boldsymbol{\zeta}_j\| u_j - \sum_{k=1, k\neq j}^{N} \left(\frac{\boldsymbol{\zeta}_{jk}^T \boldsymbol{\zeta}_k}{\|\boldsymbol{\zeta}_k\|} u_k \right). \tag{16}$$

The control laws (10a), (11a) dictate that all N agents move with bounded positive linear velocities, i.e., $0 \leq u_j < \max\{k_j\}$ for all $j \in \{1, \ldots, N\}$, which vanish only at the corresponding desired destinations $\boldsymbol{r}_{jd}$. Clearly, the motion of agents $k \neq j$ can be seen as a perturbation to the position trajectories $\boldsymbol{r}_j(t)$ of agent j. The perturbation term involving the velocities u_k for $k \neq j$ is, in general, indefinite, and depends on the agents' configurations $\boldsymbol{q}_k$ (through the vectors $\boldsymbol{\zeta}_{jk}$, $\boldsymbol{\zeta}_k$) and their destinations $\boldsymbol{r}_{kd}$ (through the control laws for u_k). Thus, $\dot{V}_j$ may be even increasing, depending on the signum of the second term including the effect of the other agents' velocities u_k. One way to deal with the issue is to use differential inequalities to establish a sufficient condition for accomplishing the objectives, as proposed in [19]. Yet, finding a suitable comparison system for bounding the solutions of the considered system is not trivial. In order to bound the time derivative of V_j let us write:

$$\dot{V}_j \leq -\|\boldsymbol{\zeta}_j\| u_j + \sum_{k=1, k\neq j}^{N} \|\boldsymbol{\zeta}_{jk}\| u_k. \tag{17}$$

Then, sufficiently away from the boundary of the constrained set $\mathcal{K}_j$ one can take:[3] $\mu_1 \in (0,1)$ and $\mu_2 > 1$ so that: $\mu_1 V_j < \|\boldsymbol{\zeta}_j\|$, $\sum_{k=1, k\neq j}^{N} \|\boldsymbol{\zeta}_{jk}\| < \mu_2 V_j$, respectively. Thus, (17) reads:

$$\dot{V}_j \leq -\mu_1 V_j k_j \tanh(\|\boldsymbol{r}_j - \boldsymbol{r}_{jd}\|) + \max_{k\neq j}\{u_k\}\mu_2 V_j.$$

Similarly, one may pick: $\nu_1 \in (0,1)$ so that: $\nu_1 \tanh(\|\boldsymbol{r}_j - \boldsymbol{r}_{jd}\|) < V_j$, to further write:

$$\dot{V}_j \leq -\mu_1 \nu_1 k_j \tanh^2(\|\boldsymbol{r}_j - \boldsymbol{r}_{jd}\|) + \max_{k\neq j}\{u_k\}\mu_2,$$

which implies that V_j is a ISS local Lyapunov function for the position error $\boldsymbol{e}_j(t) \triangleq \boldsymbol{r}_j(t) - \boldsymbol{r}_{jd}$ on a compact subset of the constrained set $\mathcal{K}_j$. This further implies that, as t increases, $\boldsymbol{e}_j(t)$ is bounded by a class $\mathcal{K}$ function of $\sup_{t>t_0}(\max_{k\neq j}\{u_k(t)\})$. Furthermore, one has out of the ISS property that if $\max_{k\neq j}\{u_k(t)\}$ converges to zero as $t \to +\infty$, so does the error $\boldsymbol{e}_j(t)$, i.e., $\boldsymbol{r}_j(t) \to \boldsymbol{r}_{jd}$ [20].

To check whether the perturbation signal $\max_{k\neq j}\{u_k(t)\}$ converges to zero as $t \to +\infty$, note first that $u_k(t)$ is by

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[3]The argument "sufficiently away from the boundary" is to ensure that $\|\boldsymbol{\zeta}_j\|$, $\|\boldsymbol{\zeta}_{jk}\|$ are bounded.

definition a class $\mathcal{K}$ function of the position error $\boldsymbol{e}_k(t) = \boldsymbol{r}_k(t) - \boldsymbol{r}_{kd}$, see (11a). Nevertheless, the dynamics of $\boldsymbol{e}_k(t)$ are governed by the signals $u_l(t)$ of the $l \neq k$ remaining agents. In other words, the ISS subsystems governing the evolution of the agents' position trajectories are feedback interconnected. This may tempt one to pursue sufficient conditions on the stability of the overall system with analysis techniques such as small-gain theorems. For such complex systems, however, such techniques in principle require non-trivial, ad-hoc bounding of the derivatives, which may be intractable or may lead to very conservative gain estimates.

Thus, for studying the convergence of the agents' trajectories with the Lyapunov-like functions considered here, let us note that the control law (11) essentially forces each agent j to perform gradient-descent along the level sets of V_j. It is expected that V_j has critical points other than the goal destination $\boldsymbol{r}_{jd}$, i.e., points $\bar{\boldsymbol{r}}_j$ at which the gradient vanishes, $\nabla V_j\big|_{\bar{\boldsymbol{r}}_j} = 0$. Here the linear velocity controller u_j (11a) does not depend on ∇V_j, but instead vanishes at the desired destination $\boldsymbol{r}_{jd}$ only; thus, system trajectories $\boldsymbol{r}_j(t)$ can not identically stay on a point where $\nabla V_j = 0$, except for the singleton $\boldsymbol{r}_{jd}$. Yet, the type of a critical point essentially dictates the behavior of the negated gradient vector field (which here is used as reference through (11b)) in the vicinity of it. A saddle point is reached by at most two streamlines in its neighborhood, while a local minimum is reached by all streamlines in its neighborhood [21]. Then, tracking the gradient vector field via (11) in the vicinity of a possible local minimum $\bar{\boldsymbol{r}}_j \neq \boldsymbol{r}_{jd}$ will trap the trajectories $\boldsymbol{r}_j(t)$ around it. In fact, it is known that the best one may achieve when resorting to gradients of scalar functions in constrained environments is to ensure that their critical points away from the desired destination are saddles, i.e., unstable equilibria; then, convergence to the goal is ensured almost everywhere, i.e., except for a set of initial conditions of measure zero. Let us now argue that the parameter δ of the Lyapunov-like function V_j essentially works in an analogous way as the tuning parameter κ of a navigation function in the Rimon-Koditschek sense [22], which implies that all undesired local minima of V_j disappear as δ increases.[4] Expecting that the parameter δ should be "large enough" is furthermore consistent with earlier results of the authors' [18] as well, where multiple objectives encoded via an approximation function similar to, *but not the same with*, the Lyapunov-like function (9) are accomplished after the parameter δ exceeds a lower bound. Obtaining sufficient lower bounds for δ in closed form is naturally depended on the characteristics of the problem at hand and is currently ongoing work.

Under this caveat, one has that for δ sufficiently large the critical points of the function V_j for each agent j are saddle

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[4]This argument is based on the fact that V_j is smooth, polar and uniformly maximal —i.e., fulfills by construction three of the four properties of a Navigation Function in the Rimon-Koditschek sense [22]— and on the similarity in the use of the δ-th and the κ-th root function, which under proper tuning renders the Morse property in the case of a Navigation Function. Nevertheless, the identification of sufficient closed forms on the bound of the parameter δ is ongoing work.

points (except for the goal destinations $\boldsymbol{r}_{jd}$), which further implies that the agents' trajectories $\boldsymbol{r}_j(t)$ converge almost everywhere to their destinations $\boldsymbol{r}_{jd}$, i.e., except for a set of initial conditions of measure zero. ∎

Remark 2: If the region O is populated with M static (circular) obstacles centered at positions $\boldsymbol{p}_{m-N}$, where $m \in \{N+1, \ldots, N+M\}$, then one can encode collision avoidance for agent $i \in \{1, \ldots, N\}$ w.r.t. the obstacle m by incorporating M squared recentered barrier functions V_{im} of the form (2) into the Lyapunov-like function (8).

Remark 3: In the case that non-mutual collision avoidance is essentially required, as in the case of an agent $j \neq 1$ avoiding the leader, then the linear velocity control gain k_j in (11a) should be selected in a careful way ensuring that the agent pursuing avoidance moves faster than the agent to be avoided. This is implied by Nagumo's necessary and sufficient conditions on the boundary of the constrained set $\mathcal{K}_j$, i.e., when the corresponding constraint becomes active, i.e., when $c_{j1} = 0$: at this point, one should ensure that $\frac{d}{dt}c_{j1} < 0$. It is easy to analytically verify that, under the angular velocity control law (11b), Nagumo's condition leads to $u_1 < u_j$, as actually expected from physical intuition.

V. SIMULATIONS

The efficacy of the proposed control algorithms is demonstrated through some representative computer simulations.

We consider a scenario of $N = 4$ agents (one leader, three followers) and $M = 1$ static obstacle in the operating environment. All agents i initiate within the region of connectivity O and need to move to their destinations $\boldsymbol{r}_{id} \in O$. Each agent knows the geometry of the obstacle (i.e., the center $\boldsymbol{p}_1$ and radius of the obstacle) and its own desired destination $\boldsymbol{r}_{id}$. The communication pattern among agents is picked as follows: Agent 1 is the leader, i.e., does not receive/take into account any information on the position of the followers. Agent 2 knows the position of agent 1, agent 3 knows the position of agents 1 and 2, and agent 4 knows the position of agents 1, 2 and 3.

The agents' initial positions are set as $\boldsymbol{r}_{1n} = \begin{bmatrix} -5 & -1 \end{bmatrix}^T$, $\boldsymbol{r}_{2n} = \begin{bmatrix} -2 & 4 \end{bmatrix}^T$, $\boldsymbol{r}_{3n} = \begin{bmatrix} -2 & -2 \end{bmatrix}^T$, $\boldsymbol{r}_{4n} = \begin{bmatrix} -2 & -4 \end{bmatrix}^T$, their destinations as $\boldsymbol{r}_{1n} = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$, $\boldsymbol{r}_{2n} = \begin{bmatrix} -4 & -3 \end{bmatrix}^T$, $\boldsymbol{r}_{3n} = \begin{bmatrix} -6 & 0 \end{bmatrix}^T$, $\boldsymbol{r}_{4n} = \begin{bmatrix} -4 & 3 \end{bmatrix}^T$, the obstacle is positioned at $\boldsymbol{p}_1 = \begin{bmatrix} -3 & -1 \end{bmatrix}^T$, the parameter δ is set equal to 1.5 and all agents' and obstacle radii are equal to .

The resulting agents' paths are illustrated in Fig. 1. The agents converge to their final destinations while always remaining in the circular region of connectivity, depicted as the external black circle. Agents 2, 4 as well as the leader 1 avoid the static obstacle, denoted as agent 5. It is also worth noting that the distributed communication pattern imposed in the scenario plausibly affects the resulting paths; for instance, agent 1 performs collision avoidance maneuvering w.r.t. the static obstacle only, agent 2 maneuvers to avoid both the static obstacle and agent 1, agent 3 maneuvers to avoid agent 1, while agent 4 has to take into account and avoid all other agents, and this results in a more complicated path.
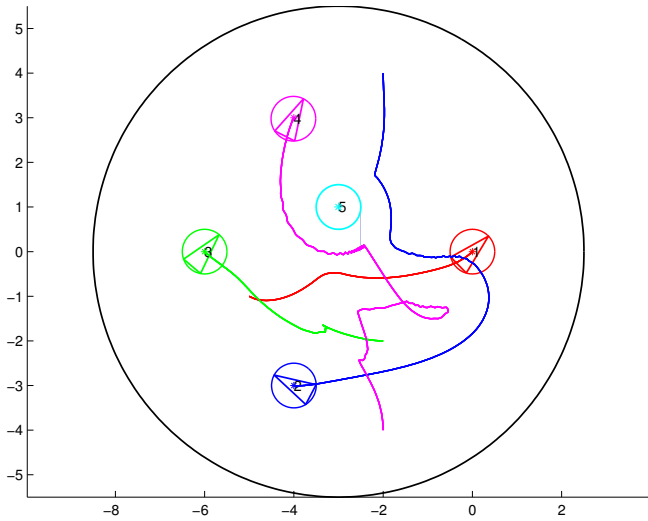
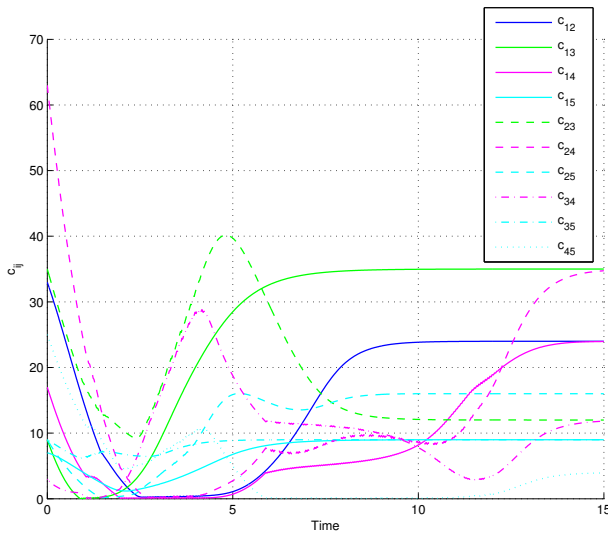Fig. 1. The resulting paths of the agents under the control laws (10), (11).

Fig. 2. The collision avoidance constraints c_{ij} remain always ≥ 0, verifying that collisions are always avoided.

To illustrate that all collisions (inter-agent and w.r.t. the static obstacle) are always avoided, Fig. 2 depicts the evolution of the collision avoidance constraints (2). The value of all constraint functions remains strictly positive, which verifies that collisions are avoided.

VI. CONCLUSIONS

This paper presented a set-theoretic formulation for multi-objective control problems encountered in the motion coordination of multiple agents, and introduced a novel class of Lyapunov-like barrier functions to encode the agents' constrained sets. The proposed Lyapunov-like functions encode objectives such as collision avoidance, proximity and convergence to desired destinations, and are amenable to distributed control formulations and Lyapunov-based control design and analysis. The efficacy of the approach and its relevance to surveillance missions using multiple nonholonomic vehicles

is illustrated via non-trivial simulations. Our current work focuses on the use of Lyapunov-like barrier functions to encode vision-based sensing constraints, which are pertinent to surveillance and coverage problems with robotic vehicles.

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