Control of nonholonomic systems using reference vector fields

Dimitra Panagou, Herbert G. Tanner and Kostas J. Kyriakopoulos

***Abstract*— This paper presents a control design methodology for n-dimensional nonholonomic systems. The main idea is that, given a nonholonomic system subject to κ Pfaffian constraints, one can define a smooth, N-dimensional reference vector field F, which is nonsingular everywhere except for a submanifold containing the origin. The dimension $\mathrm{N} \leq n$ of F depends on the structure of the constraint equations, which induces a foliation of the configuration space. This foliation, together with the objective of having the system vector field aligned with F, suggests a choice of Lyapunov-like functions V. The proposed approach recasts the original nonholonomic control problem into a lower-dimensional output regulation problem, which although nontrivial, can more easily be tackled with existing design and analysis tools. The methodology applies to a wide class of nonholonomic systems, and its efficacy is demonstrated through numerical simulations for the cases of the unicycle and the n-dimensional chained systems, for $n = 3, 4$.**

I. INTRODUCTION

Control of nonholonomic systems has been a field of rigorous research, motivated by both theoretical and practical considerations. From a practical viewpoint, nonholonomic systems model a wide class of mechanical systems, bringing thus the need for methodologies addressing stabilization, path, and trajectory tracking problems. From a theoretical viewpoint, Brockett's condition [1] and the results in [2] have established that nonholonomic systems can not be asymptotically stabilized to a single equilibrium using smooth, or even continuous, time-invariant feedback. To overcome this limitation, research has focused on solutions that can be broadly classified into two groups, those that employ time-varying feedback, either smooth [3]–[7], or non-smooth with respect to (w.r.t.) the state [8]–[11], and those that use time-invariant, non-smooth state feedback. The latter approach includes piecewise continuous [2], [12], discontinuous [13]–[19], and hybrid/switching control solutions [20]–[23].

Among the variety of nonholonomic systems, the class of n-dimensional chained systems has received special attention, in part because they model the kinematics of underactuated mechanical systems, for instance of unicycle or car-like mobile robots pulling trailers. Numerous control solutions have been proposed, either smooth and time-varying which yield slow convergence, or non-smooth which yield fast (exponential) convergence; for this reason, the latter solutions are preferable in practical aspects. In the latter case, the control design often employs nonlinear state transformations [13], [15], [18], [19], [24]–[26] and the control laws are extracted in the new coordinate system, using either linear or nonlinear techniques. However, the coordinate transformations are not always straightforward and thus the derivation of the control laws in general remains non-trivial.

This paper provides a framework for the construction of such control laws, building on the geometric generalization of one of our prior control designs. The considered nonholonomic systems fall into the general class of drift-free systems

$$\dot{\boldsymbol{q}} = \sum_{i=1}^{m} \mathbf{g}_i(\boldsymbol{q}) u_i, \tag{1}$$

where $\boldsymbol{q} \in \mathcal{C}$ is the configuration (state) vector, $\mathcal{C}$ is the configuration space (an n-dimensional smooth manifold), u_i are the control inputs, $\mathbf{g}_i(\boldsymbol{q})$ are the control vector fields, $i = 1, \ldots, m$, whereas the considered nonholonomic Pfaffian constraints are of the form

$$\boldsymbol{A}(\boldsymbol{q})\dot{\boldsymbol{q}} = \mathbf{0}, \tag{2}$$

with $\boldsymbol{A}(\boldsymbol{q}) \in \mathbb{R}^{\kappa \times n}$. The main idea of the approach is that one can define a smooth, N-dimensional reference vector field $\mathbf{F}(\cdot)$, given by a family of vector fields with certain, desired properties (see Section II, II-A). The dimension $\mathrm{N} \leq n$ and the analytic form of the vector field $\mathbf{F}$ are specified by the explicit form (2) of the constraints in the following sense: depending on the structure of $\boldsymbol{A}(\boldsymbol{q})$, the configuration space $\mathcal{C}$ is trivially decomposed into $\mathcal{F} = \mathcal{L} \times \mathcal{T}$, where $\mathcal{L}$ is the "leaf" space, $\mathcal{T}$ is the "fiber" space, $\dim\mathcal{L} = \mathrm{N}$, satisfying $n = \dim\mathcal{L} + \dim\mathcal{T}$. In the sequel, the local coordinates $\boldsymbol{x} \in \mathbb{R}^{\mathrm{N}}$ on the leaf are called *leafwise states* and the local coordinates $\boldsymbol{t} \in \mathbb{R}^{n-\mathrm{N}}$ on the fiber are called *transverse states*. The vector field $\mathbf{F}(\cdot)$ is defined tangent to $\mathcal{L}$ in terms of the leafwise states $\boldsymbol{x}$, and is non-vanishing everywhere on $\mathcal{L}$ except for the origin $\boldsymbol{x} = \mathbf{0}$ of the local coordinate system. For each $\boldsymbol{t} \in \mathcal{T}$, all integral curves of $\mathbf{F}(\cdot)$ contain the origin $\boldsymbol{x} = \mathbf{0}$ of the coordinate system on the leaf that corresponds to $\boldsymbol{t}$. As a consequence of defining $\mathrm{N} < n$, $\mathbf{F}(\cdot)$ is singular (i.e. vanishes) on a submanifold $\mathcal{A}$ that contains the origin; this singularity may necessitate switching for initial conditions $\boldsymbol{q_i} \in \mathcal{A}$. Away from the singularity submanifold, $\mathbf{F}$ serves as a velocity reference for (1). This means that, at each $\boldsymbol{q} \in \mathcal{C}$, the system vector field $\dot{\boldsymbol{q}} \in T_{\boldsymbol{q}}\mathcal{C}$ is steered to be made parallel to the vector field $\mathbf{F}(\cdot)$. This in turn implies that the constraint equations (2) take the form $\boldsymbol{A}(\boldsymbol{q})\mathbf{F}(\boldsymbol{q}) = \mathbf{0}$; we say in this case that $\mathbf{F}(\boldsymbol{q})$ satisfies, or is consistent with, the constraints at $\boldsymbol{q} \in \mathcal{C}$ (see Section II). In this sense, one can use the available control authority to steer the system vector field into the tangent

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bundle of the integral curves of $\mathbf{F}$, and "flow" in the direction of the reference vector field on its way to the origin. In the sections that follow we show that these two objectives dictate the choice of particular functions V, and enable one to establish convergence based on standard techniques.

In relation to the authors' prior work [27], this paper views some of the earlier results from the new geometric perspective and suggests new control laws. The framework presented here allows the extension of the methodology to systems with $\kappa > 1$ Pfaffian constraints. As a case study, we treat the case of n-dimensional chained systems, $n \geq 3$, which are subject to $\kappa = n - 2$ constraint equations. Also, a new class of N-dimensional vector fields $\mathbf{F}$ is adopted, which is of simpler analytic form compared to those in [27].

With respect to existing nonholonomic control methods, the novelty of this approach is that it recasts the original problem of steering the state to the origin into a lower-dimensional output regulation problem, and although it preserves the nonholonomic nature of the original problem, it offers a more favorable ratio of inputs versus states, and allows a uniform stability analysis with standard tools. The new geometric perspective exposes the interdependence of the state variables and highlights a potential time-scale decomposition, which permits the use of additional analysis techniques, such as those related to singular perturbations and slowly varying systems. In view of Brockett's condition, the solutions cannot be stable in the Lyapunov sense.

The paper is organized as follows: Section II presents the construction of the reference vector fields $\mathbf{F}$ and the control design for the unicycle, which is an example of systems with a single Pfaffian constraint. Section III illustrates how the proposed framework extends to n-dimensional chained systems, where $\kappa \geq 1$ constraints apply. Our conclusions and plans for future extensions are given in Section IV.

II. Construction of the Vector Fields

A vector field $\mathbf{F}(\cdot) : \mathcal{C} \to T\mathcal{C}$ is said to be *consistent* with the nonholonomic constraints (2) at a point $\boldsymbol{q} \in \mathcal{C}$, or that it satisfies the *consistency* condition at $\boldsymbol{q}$, if it fulfils the constraint equations at $\boldsymbol{q}$, i.e. if

$$\boldsymbol{A}(\boldsymbol{q})\mathbf{F}(\boldsymbol{q}) = \mathbf{0}. \tag{3}$$

Since $\boldsymbol{A}(\boldsymbol{q}) \in \mathbb{R}^{\kappa\times n}$, it follows that in some local coordinates, the vector field $\mathbf{F} = \sum_{j=1}^{n} \mathrm{F}_j \frac{\partial}{\partial q_j}$, where $\left\{\frac{\partial}{\partial q_1}, \ldots, \frac{\partial}{\partial q_n}\right\}$ are the unit basis vectors of the tangent space $T_{\boldsymbol{q}}\mathcal{C}$, is in the representation of an n-dimensional vector-valued map. The explicit form of the condition (3) affects the analytic form of $\mathbf{F}$. To see how, consider the resulting linear (in terms of F_j) system:

$$\begin{aligned} a_{11}\,\mathrm{F}_1 + a_{12}\,\mathrm{F}_2 + \ldots + a_{1n}\,\mathrm{F}_n &= 0, \\ &\vdots \\ a_{\kappa 1}\,\mathrm{F}_1 + a_{\kappa 2}\,\mathrm{F}_2 + \ldots + a_{\kappa n}\,\mathrm{F}_n &= 0; \end{aligned}$$

then, if for example $\boldsymbol{A}(\boldsymbol{q})$ contains one zero column, i.e. if $\begin{bmatrix} a_{1j}(\boldsymbol{q}) & \ldots & a_{\kappa j}(\boldsymbol{q}) \end{bmatrix}^{\mathrm{T}} = \mathbf{0}$ for some $j \in \{1, \ldots, n\}$, the corresponding component F_j of the vector field does not affect whether the constraints are satisfied or not, because the linear map always sends F_j to zero. One could therefore define a vector field $\mathbf{F}$ in which $\mathrm{F}_j = 0$. Thus, since the reference vector field does not specify any motion along q_j, it may as well be independent of this variable, and be defined as an $\mathrm{N} = (n-1)$ dimensional vector field. In general, if $\boldsymbol{A}(\boldsymbol{q})$ has $0 \leq n_0 < n$ zero columns, the dimension of the vector field $\mathbf{F}$ can be $\mathrm{N} = n - n_0$. Note, however, that this simplification comes at a cost: dropping some of the state variables from the definition of $\mathbf{F}$ permits the latter to be singular in a whole submanifold which contains the origin, and forces the designer to use switching control for the cases when the system is initiated on this submanifold.

The control strategy we consider can be summarized as follows: Given (1) subject to (2):

1) find an N-dimensional vector field $\mathbf{F}(\cdot) : \mathcal{L} \to T\mathcal{L}$, the integral curves of which contain the origin $\boldsymbol{x} = \mathbf{0}$ of the local coordinate system, and

2) design a feedback control scheme to align the system vector field $\dot{\boldsymbol{q}} \in T_{\boldsymbol{q}}\mathcal{C}$ with $\mathbf{F}$, and "flow" along $\mathbf{F}$ ensuring that $\dot{\boldsymbol{q}}$ is non-vanishing everywhere but the origin $\boldsymbol{q} = \mathbf{0}$.

A. The unicycle: A first example

To illustrate the strategy, consider the unicycle, given by

$$\dot{\boldsymbol{q}} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \end{bmatrix}^{\mathrm{T}} u_1 + \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}} u_2, \tag{4}$$

where $\boldsymbol{q} = \begin{bmatrix} x & y & \theta \end{bmatrix}^{\mathrm{T}} \in \mathcal{C}$ is the configuration vector, $\mathcal{C}$ is the configuration space, x, y, θ are the generalized coordinates, with x, y being the position coordinates and θ the orientation w.r.t. a global cartesian coordinate frame $\mathcal{G}$, and u_1, u_2 are the control inputs. The $\kappa = 1$ nonholonomic constraint is written in Pfaffian form as

$$\begin{bmatrix} -\sin\theta & \cos\theta & 0 \end{bmatrix} \dot{\boldsymbol{q}} = 0 \;\Leftrightarrow\; \langle \boldsymbol{a}^{\mathrm{T}}(\boldsymbol{q}), \dot{\boldsymbol{q}} \rangle = 0,$$

where $\langle\cdot,\cdot\rangle$ stands for the inner product. For a vector field $\mathbf{F} = \mathrm{F}_x \frac{\partial}{\partial x} + \mathrm{F}_y \frac{\partial}{\partial y} + \mathrm{F}_\theta \frac{\partial}{\partial \theta}$ to satisfy the consistency condition (3), it should satisfy

$$\underbrace{\begin{bmatrix} -\sin\theta & \cos\theta & 0 \end{bmatrix}}_{\boldsymbol{a}^{\mathrm{T}}(\boldsymbol{q})} \begin{bmatrix} \mathrm{F}_x \\ \mathrm{F}_y \\ \mathrm{F}_\theta \end{bmatrix} = 0 \Rightarrow \mathrm{F}_y \cos\theta - \mathrm{F}_x \sin\theta = 0. \tag{5}$$

In this case, the constraint vector $\boldsymbol{a}^{\mathrm{T}}(\boldsymbol{q})$ contains $n_0 = 1$ zero element and thus the component F_θ does not affect whether the consistency condition (5) is satisfied. One can define $\mathrm{F}_\theta = 0$, and search for an $\mathrm{N} = n - n_0 = 2$ dimensional vector field $\mathbf{F}(\cdot)$, in terms of F_x, F_y only. From a geometric point of view, setting $\mathrm{F}_\theta = 0$ implies that, for each $\boldsymbol{q} \in \mathcal{C}$, the vector field $\mathbf{F}(\cdot)$ should lie in the subspace $\mathcal{W} = \left\{\boldsymbol{w} \in T_{\boldsymbol{q}}\mathcal{C} \mid \boldsymbol{w} = \begin{bmatrix} w_x & w_y & 0 \end{bmatrix}^{\mathrm{T}}\right\}$ of the tangent space $T_{\boldsymbol{q}}\mathcal{C}$ of $\mathcal{C}$. Thus, the vector field $\mathbf{F}(\cdot)$ should be tangent to the submanifold $\mathcal{S} = \mathbb{R}^2$. The submanifold $\mathcal{S}$ is a 2-dimensional leaf of a codimension-1 foliation $\mathcal{F} = \mathcal{L} \times \mathcal{T}$ of the configuration space into $\mathcal{C} = \mathbb{R}^2 \times \mathbb{S}^1$ (Fig. 1). Consequently, the vector field $\mathbf{F}(\cdot)$ is tangent to the leafwise directions $\mathcal{L} = \mathbb{R}^2$ of $\mathcal{F}$.

Let us now find an analytic expression of the vector field $\mathbf{F}(\cdot) : \mathcal{C} \to \mathbb{R}^2$, i.e. of the components F_x, F_y, based on

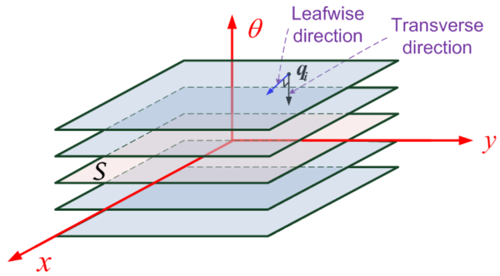
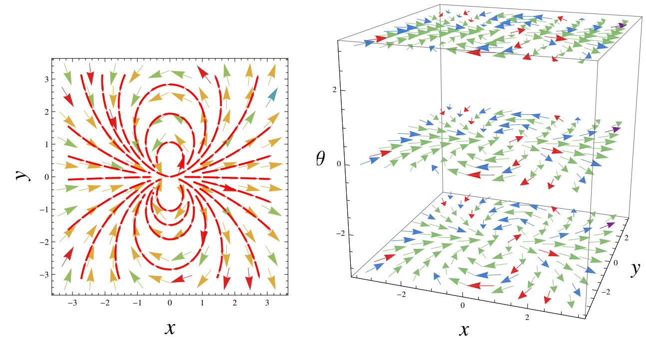


Fig. 1. The foliation $\mathcal{F}$ of the 3-dimensional configuration space $\mathcal{C}$ of the unicycle into $\mathcal{C} = \mathbb{R}^2 \times \mathbb{S}^1$



(a) The vector field $\mathbf{F}(\boldsymbol{r})$ on $\mathbb{R}^2$ (b) The vector field $\mathbf{F}(\boldsymbol{r})$ on $\mathcal{C}$

Fig. 2. The vector field $\mathbf{F}(\boldsymbol{r})$ in the case of the unicycle.

the consistency condition (5). For each $\theta \in \mathbb{S}^1$, the condition (5) is a linear equation in F_x and F_y. We can pick $\mathrm{F}_x = \|\mathbf{F}\| \cos\phi$ and $\mathrm{F}_y = \|\mathbf{F}\| \sin\phi$, where $\|\mathbf{F}\|$ is the Euclidean norm of the vector field at $\boldsymbol{q} \in \mathcal{C}$, and ϕ is the direction of the vector $\mathbf{F}(\boldsymbol{q})$ w.r.t. a global frame $\mathcal{G}$. The consistency condition (5) then becomes

$$\langle \boldsymbol{a}^{\mathrm{T}}(\boldsymbol{q}), \mathbf{F} \rangle = \|\mathbf{F}\| \sin(\phi - \theta) = 0. \tag{6}$$

For a nonsingular vector field $\mathbf{F}$, (6) implies that $\sin(\phi - \theta) = 0 \Rightarrow \phi - \theta = \xi\pi, \xi \in \mathbb{Z}$. Then, for $\boldsymbol{q} \to \mathbf{0}$, (6) reads $\theta \to 0 \Rightarrow \phi \to \xi\pi, \xi \in \mathbb{Z}$; i.e. the direction ϕ of the nonsingular vector field $\mathbf{F}$ should converge to either $\phi = 0$ or $\phi = \pm\pi$, as $\boldsymbol{q} \to \mathbf{0}$.

This requirement justifies the choice of the 2-dimensional vector field $\mathbf{E}(\boldsymbol{r})$ of the electric point dipole as a reference vector field for the case of the unicycle.[1] In [27], the vector field $\mathbf{E}(\boldsymbol{r})$ in a workspace $\Omega \subseteq \mathbb{R}^2$ was approximated as

$$\mathbf{F}(\boldsymbol{r}) = \lambda \left(\boldsymbol{p}^{\mathrm{T}} \boldsymbol{r}\right) \boldsymbol{r} - \boldsymbol{p} + \boldsymbol{p}\, e^{-\|\boldsymbol{r}\|^2}, \tag{7}$$

where $\lambda > 2$, $\boldsymbol{r} = \begin{bmatrix} x & y \end{bmatrix}^{\mathrm{T}}$, $\boldsymbol{p} \in \mathbb{R}^2$. The vector $\boldsymbol{p}$ is selected depending on the structure of $\boldsymbol{a}(\boldsymbol{q})$ at the origin: $\boldsymbol{p}$ is required to lie on the (local manifestation of the) constraint surface at the origin, in order to be consistent with the constraints:

$$\langle \boldsymbol{a}^{\mathrm{T}}(\mathbf{0}), \boldsymbol{p} \rangle = 0 \Rightarrow \begin{bmatrix} -\sin(0) & \cos(0) & 0 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_\theta \end{bmatrix} = 0, \tag{8}$$

where $p_\theta = 0$, for the same reason that F_θ can be set to zero. The condition (8) is satisfied for $p_x \in \mathbb{R}$ and $p_y = 0$, and since $\boldsymbol{p}$ should be non-zero, we set $\boldsymbol{p} \triangleq \begin{bmatrix} p_x & 0 \end{bmatrix}^{\mathrm{T}}$ with $p_x \neq 0$. The vector field components F_x, F_y in (7) read

$$\mathrm{F}_x = \lambda p_x x^2 - p_x + p_x e^{-(x^2+y^2)}, \quad \mathrm{F}_y = \lambda p_x x y, \tag{9}$$

where $\lambda > 2$. In (9), the exponential $e^{-\|\boldsymbol{r}\|^2}$ complicates the derivation of control laws, and motivates a slight modification of $\mathbf{E}(\boldsymbol{r})$, in the form

$$\mathrm{F}_x = 3p_x x^2 - p_x(x^2 + y^2), \quad \mathrm{F}_y = 3p_x x y. \tag{10}$$

The resulting vector field is shown in Fig. 2(a). The vector field (10) has the same desirable properties as (9), namely, has integral lines which contain the origin $\boldsymbol{r} = \mathbf{0}$ and satisfy (6) there, yet it is much simpler compared to (9). It is easy to verify that (10) is given by

$$\mathbf{F}(\boldsymbol{r}) = \lambda \left(\boldsymbol{p}^{\mathrm{T}} \boldsymbol{r}\right) \boldsymbol{r} - \boldsymbol{p} \left(\boldsymbol{r}^{\mathrm{T}} \boldsymbol{r}\right), \tag{11}$$

for $\lambda = 3$. As expected, the dimension $\mathrm{N} = 2$ of the vector field is the dimension of the leafwise direction $\mathcal{L} = \mathbb{R}^2$, and the vector field is in terms of the leafwise states x, y. Inspired by this, we propose the following general class of N-dimensional vector fields

$$\mathbf{F}(\boldsymbol{x}) = \lambda \left(\boldsymbol{p}^{\mathrm{T}} \boldsymbol{x}\right) \boldsymbol{x} - \boldsymbol{p} \left(\boldsymbol{x}^{\mathrm{T}} \boldsymbol{x}\right), \tag{12}$$

where $\mathrm{N} \leq n$, $\boldsymbol{x} \in \mathbb{R}^{\mathrm{N}}$ is the vector of the leafwise states, $\boldsymbol{p} \in \mathbb{R}^{\mathrm{N}}$ and $\lambda \geq 2$, which we call the *N-polar vector fields*. In the following sections, we show that (12) can be used to design controllers for a wide class of nonholonomic systems.

Going back to the unicycle, since $\mathrm{F}_\theta = 0$, the vector field (10) on $\mathcal{C}$ does not vary along the transverse direction of the foliation $\mathcal{F}$ (Fig. 2(b)).

To enable the alignment of the system's vector field with $\mathbf{F}$, we define the map $h(\cdot) : \mathcal{C} \to \mathbb{R}$

$$h(\boldsymbol{q}) = \langle \boldsymbol{a}^{\mathrm{T}}(\boldsymbol{q}), \mathbf{F} \rangle. \tag{13}$$

If $h(\boldsymbol{q}) = 0$, $\mathbf{F}$ locally belongs to the null space of the constraint co-vector $\boldsymbol{a}^{\mathrm{T}}(\boldsymbol{q})$, and can therefore be realized locally as a linear combination of the control vector fields $\mathbf{g}_i(\boldsymbol{q})$. Thus we can treat $h(\boldsymbol{q}) \neq 0$ as an error variable, or output, which should be regulated to zero. For a nonsingular vector field $\mathbf{F}$, $h(\boldsymbol{q}) = 0 \overset{(6)}{\Leftrightarrow} \{\theta - \phi = 0 \;\text{ or }\; \theta - \phi = \pm\pi\}$. In this case, the orientation θ of the unicycle is tangent to the integral line of the vector field (10).

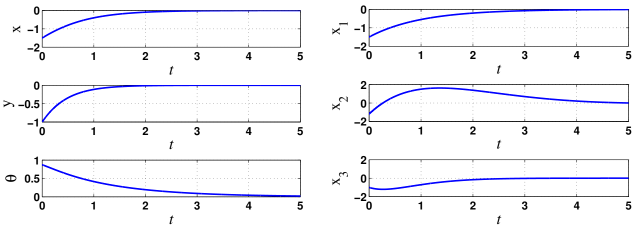
Getting $h(\boldsymbol{q}) \to 0$ in the case of the unicycle is realized by making $\theta \to \phi + \xi\pi,\ \xi \in \mathbb{Z}$. Define the consistency error $s = \theta - \phi$ and force the dynamics $\dot{s} = -ks,\ k > 0$ on it, by choosing u_2 as

$$\dot{\theta} - \dot{\phi} = -k(\theta - \phi) \overset{(4)}{\Rightarrow} u_2 = -k(\theta - \phi) + \dot{\phi}, \tag{14}$$

where $\dot{\phi} = \frac{(3y\,\mathrm{F}_x - 4x\,\mathrm{F}_y)\cos\theta + (3x\,\mathrm{F}_x + 2y\,\mathrm{F}_y)\sin\theta}{\mathrm{F}_x{}^2 + \mathrm{F}_y{}^2}\, u_1$.

The quantity $\dot{\phi}$ is not defined when $\|\mathbf{F}\| = 0$, i.e. at the singular points of $\mathbf{F}$ on the submanifold $\mathcal{A} = \{\boldsymbol{q} \in$

[1]The flow lines of $\mathbf{E}(r, \varphi) = \frac{2p\cos\varphi}{4\pi\epsilon_0 r^3}\hat{\boldsymbol{r}} + \frac{p\sin\varphi}{4\pi\epsilon_0 r^3}\hat{\boldsymbol{\varphi}}$ are given by $r = r_0 \sin^2(\varphi - \varphi_1)$, $r_0 > 0$, where (r, φ) are the polar coordinates of the position vector $\boldsymbol{r} = \begin{bmatrix} x & y \end{bmatrix}^{\mathrm{T}}$ and (p, φ_1) are the polar coordinates of the dipole moment $\boldsymbol{p} \in \mathbb{R}^2$. Set $\varphi_1 = 0$, i.e. take $\boldsymbol{p} = \begin{bmatrix} p_x & 0 \end{bmatrix}^{\mathrm{T}}$, $p_x \neq 0$, then $r \to 0$ implies that $\sin\varphi \to 0 \Rightarrow \varphi \to \xi\pi, \xi \in \mathbb{Z}$.



(a) Unicycle trajectories (b) 3-d chained trajectories

Fig. 3. System response for the unicycle and the 3-d chained system

$\mathcal{C} \mid \boldsymbol{q} = \begin{bmatrix} 0 & 0 & \theta \end{bmatrix}^{\mathrm{T}}\}, \theta \in \mathbb{S}^1$. Thus if $\boldsymbol{q_i} \in \mathcal{A}$, switching to a different control law should occur, for instance to $u_2 = -k\theta$.

The conditions (13), (14) determine the motion of the system in the transverse direction. With the choice of (14) the unicycle aligns itself with $\mathbf{F}$ as it moves *from leaf to leaf*. Along the leaves, on the other hand, the system should be driven to the origin of each local (x, y) coordinate system. In order to analyze the dynamics on the leaves, we consider a continuously differentiable function V in terms of the leafwise states x, y and the consistency error s

$$V = \frac{1}{2}(x^2 + y^2 + s^2) = \frac{1}{2}\left(x^2 + y^2 + (\theta - \phi)^2\right),$$

and take its time derivative along the system trajectories as $\dot{V} \overset{(14)}{=} (x\cos\theta + y\sin\theta)u_1 - k(\theta - \phi)^2$. Then, choosing the control input u_1 as $u_1 = -k_1 \operatorname{sgn}(x\cos\theta + y\sin\theta)\|\boldsymbol{r}\|$, $k_1 > 0$, where $\operatorname{sgn}(a) = 1$ for $a \geq 0$, and $\operatorname{sgn}(a) = -1$ for $a < 0$, yields $\dot{V} = -k_1(x\cos\theta + y\sin\theta)\operatorname{sgn}(x\cos\theta + y\sin\theta)\|\boldsymbol{r}\| - k(\theta - \phi)^2 \leq 0$. Then one has $\dot{V}(\mathbf{0}) = 0$, since $\phi\big|_{x=0,y=0} = 0$. According to LaSalle's invariance principle [28], and given that V is positive definite, the system trajectories converge to the largest invariant set contained in the set $\Omega = \{\boldsymbol{q} \in \mathcal{C} \mid \dot{V}(\boldsymbol{q}) = 0\}$. The set Ω is given as $\Omega = \Omega_1 \vee \Omega_2$, where $\Omega_1 = \{\boldsymbol{q} \in \mathcal{C} \mid \{x\cos\theta + y\sin\theta = 0\} \wedge \{\theta = \phi\}\}$ and $\Omega_2 = \{\boldsymbol{q} \in \mathcal{C} \mid \{x = y = 0\} \vee \{\theta = \phi\}\}$. After some algebra, one gets $\Omega_1 = \{\boldsymbol{q} \in \mathcal{C} \mid \{x = 0\} \wedge \{\theta = \pi\}\}$, and $\Omega_2 = \{\boldsymbol{q} = \mathbf{0}\}$. One can easily verify that Ω_1 is not an invariant set, since for $\boldsymbol{q_i} = \begin{bmatrix} 0 & y & \pi \end{bmatrix}^{\mathrm{T}}$ the system has $u_1 \neq 0$ and thus escapes Ω_1, whereas Ω_2 is an invariant set. Consequently, the largest invariant set reduces to the origin, and thus the system trajectories converge to $\Omega_2 = \{\boldsymbol{q} = \mathbf{0}\}$. The closed-loop trajectories are depicted in Fig. 3(a).

III. Chained Systems

Consider the n-dimensional chained system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ x_2 \\ \vdots \\ x_{n-1} \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u_2, \tag{15}$$

where $\boldsymbol{q} = \begin{bmatrix} x_1 & x_2 & x_3 & \dots & x_n \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^n$ the state vector and u_1, u_2 the control inputs. The system is subject to $\kappa = n - 2$ nonholonomic constraints, written in Pfaffian form as

$$\begin{bmatrix} -x_2 & 0 & 1 & 0 & \dots & 0 \\ -x_3 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -x_{n-1} & 0 & 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{16}$$

where $\boldsymbol{A}(\boldsymbol{q}) \in \mathbb{R}^{(n-2)\times n}$. The constraint matrix $\boldsymbol{A}(\boldsymbol{q})$ has $n_0 = 1$ zero column, which is associated with the generalized coordinate x_2. In this case, one can define $\mathrm{F}_{x_2} = 0$, and look for an $\mathrm{N} = (n-1)$ dimensional vector field $\mathbf{F}(\cdot)$, in terms of F_{x_j}, where $j \in \{1, 3, \dots, n\}$.

Thus, the configuration space $\mathbb{R}^n$ is foliated into $\mathbb{R}^{\mathrm{N}} \times \mathbb{R}$, where $\begin{bmatrix} x_1 & x_3 & \dots & x_n \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{\mathrm{N}}$ are the leafwise states and $x_2 \in \mathbb{R}$ is the transverse state. The vector $\boldsymbol{p} \in \mathbb{R}^{\mathrm{N}}$ should satisfy the constraints at the origin,

$$\underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{bmatrix}}_{\boldsymbol{A}(\mathbf{0}) \in \mathbb{R}^{\kappa \times n}} \begin{bmatrix} p_1 \\ 0 \\ p_3 \\ \vdots \\ p_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \Rightarrow \begin{array}{c} p_1 \neq 0, \\ p_3 = \ldots = p_n = 0 \end{array}$$

where by definition $p_2 = 0$. Take $\boldsymbol{p} = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{\mathrm{N}}$ and $\lambda = 3$, then (12) yields

$$\begin{aligned} &\mathrm{F}_{x_1} = 2{x_1}^2 - {x_3}^2 - {x_4}^2 - \ldots - {x_n}^2, \\ &\mathrm{F}_{x_3} = 3x_1x_3,\ \mathrm{F}_{x_4} = 3x_1x_4,\ \ldots,\ \mathrm{F}_{x_n} = 3x_1x_n. \end{aligned} \tag{17}$$

Define the $\kappa = (n-2)$ maps $h_k(\cdot) : \mathbb{R}^n \to \mathbb{R}$ as $h_k(\boldsymbol{q}) = \langle \boldsymbol{a_k}^{\mathrm{T}}(\boldsymbol{q}), \mathbf{F} \rangle$, where $\boldsymbol{a_k}^{\mathrm{T}}(\boldsymbol{q})$, $k = 1, \dots, \kappa$ are the constraint vectors. This results to

$$h_1(\boldsymbol{q}) = -x_2\,\mathrm{F}_{x_1} + \mathrm{F}_{x_3},\ \ \ldots\ \ , h_\kappa(\boldsymbol{q}) = -x_{n-1}\,\mathrm{F}_{x_1} + \mathrm{F}_{x_n}.$$

Each $h_k(\boldsymbol{q}) \neq 0$ is an output which should be regulated to zero. This can be achieved if $x_{n-1} \to \frac{\mathrm{F}_{x_n}}{\mathrm{F}_{x_1}}$, $\forall n \geq 3$.

A. The $n = 3$ dimensional chained system

The derivation of the control laws u_1, u_2 can be simplified if one considers the analytic expressions of the outputs $h_k(\boldsymbol{q})$. Take for instance the $n = 3$ dimensional chained system, and the corresponding $\kappa = n - 2 = 1$ map

$$h_1(\boldsymbol{q}) = -3x_1(x_1x_2 - x_3) + x_2({x_1}^2 + {x_3}^2).$$

One can require that each one of the terms of h_1 converges to zero. This occurs if, for instance, $s_1 \triangleq x_1x_2 - x_3 \to 0$ and $s_2 \triangleq {x_1}^2 + {x_3}^2 \to 0$. Then, one can take a continuously differentiable function in terms of the errors s_1, s_2 and the leafwise states x_1, x_3 as

$$V = \frac{1}{2}\left(x_1x_2 - x_3\right)^2 + \frac{1}{2}\left({x_1}^2 + {x_3}^2\right).$$

The time derivative of V along the system trajectories is

$$\dot{V} \overset{(15)}{=} x_1(x_1x_2 - x_3)u_2 + (x_1 + x_3x_2)u_1. \tag{18}$$

We would like to render $\dot{V}$ negative semi-definite, and also render the origin the largest invariant set contained in the set $\Omega = \{\boldsymbol{q} \in \mathbb{R}^3 \mid \dot{V} = 0\}$. Then, convergence of the system trajectories to the origin can be established via LaSalle's invariance principle, with the following caveat: the use of the invariance principle requires that either V is radially unbounded, or that the level surfaces of V are compact sets. In this case neither of the two conditions are automatically satisfied, since for $x_1 = x_3 = 0,\ |x_2| \to \infty \Rightarrow V = 0$. However, V is in fact radially unbounded as long as $x_1 \neq 0$; then, if $x_1(0) \neq 0$ and the system is controlled so that x_1 maintains its sign and varies at a much slower time scale compared to $x_2,\ x_3$, then the invariance principle applies.[2] Thus, if x_1 is assumed nonzero along the trajectories of (15), the following positive definite function can be used

$$V_1 = \frac{1}{2}\left(x_1 x_2 - x_3\right)^2 + \frac{1}{2}{x_3}^2,$$

whose time derivative along the system trajectories is

$$\dot{V}_1 \overset{(15)}{=} x_1(x_1 x_2 - x_3)u_2 + x_3 x_2 u_1. \tag{19}$$

Then, if the control inputs u_1, u_2 are chosen so that the state x_1 converges to zero slowly, and also

$$\dot{V}_1 = -k_2(x_1 x_2 - x_3)^2 - k_3 {x_3}^2, \tag{20}$$

for $x_1 \neq 0$ and $k_2, k_3 > 0$, the set where $\dot{V}_1$ vanishes is $\Omega_1 = \{\boldsymbol{q}|\{x_1 x_2 = x_3\} \wedge \{x_3 = 0\}\} \Rightarrow \Omega_1 = \{\boldsymbol{q}|x_2 = x_3 = 0\}$.

In order to design the control laws so that the above analysis applies, one can first choose $u_1 = -k_1 x_1$, where k_1 is a small positive scalar, to directly control the convergence rate of the state x_1. Then, combining (19), (20), one has

$$u_2 = -k_2(x_2 - \frac{x_3}{x_1}) + \frac{k_1 x_3(x_1 x_2 - \frac{k_3}{k_1}x_3)}{x_1(x_1 x_2 - x_3)}. \tag{21}$$

If one selects $k_3 = k_1$, (21) is further simplified to

$$u_2 = -k_2 x_2 + (k_2 + k_1)\frac{x_3}{x_1},\ \text{for } x_1 \neq 0. \tag{22}$$

For $x_1(0) \neq 0$, ensuring that x_1 converges to zero slower than x_2, one can have $\boldsymbol{q}(t)$ converge to the set Ω_1. To tune the gains $k_1,\ k_2$ so that this condition applies, consider that the derivative (20) for $k_3 = k_1$ reads: $\dot{V}_1 \leq -2\min\{k_2, k_1\}V_1$, while the dynamics of x_1 read: $\dot{x}_1 = -k_1 x_1$; picking $k_2 > k_1$ implies that V_1 vanishes at least twice faster than x_1.

Finally, if $x_1(0) = 0$ one needs to switch to a different control strategy to drive the system away from the $x_1 = 0$ surface, and then apply the scheme described above [30]; one option is $u_1(t) \neq 0,\ u_2 = 0$ for $t < T,\ T > 0$. The closed-loop system trajectories of the 3-dimensional chained system for $k_1 = 1,\ k_2 = 2.5$ are shown in Fig. 3(b).

B. The $n > 3$ dimensional chained system

The same guidelines can be applied to systems with $\kappa > 1$ constraint equations. In order to illustrate this, consider the $n = 4$ dimensional chained system, and the $\mathrm{N} = n - 1 = 3$ dimensional vector $\mathbf{F}(\cdot) = \mathrm{F}_{x_1}\frac{\partial}{\partial x_1} + \mathrm{F}_{x_3}\frac{\partial}{\partial x_3} + \mathrm{F}_{x_4}\frac{\partial}{\partial x_4}$,

$$\mathrm{F}_{x_1} = 2{x_1}^2 - {x_3}^2 - {x_4}^2,\ \mathrm{F}_{x_3} = 3x_1 x_3,\ \mathrm{F}_{x_4} = 3x_1 x_4.$$

[2]Note that the requirement on a slowly-convergent x_1 is frequently used in the literature of chained systems [19], [29]. With this insight, we later introduce a time-scale decomposition for the whole class of chained systems which overcomes this limitation of the invariance principle and can bring the analysis of these systems under a common framework.

Then, the corresponding $\kappa = n - 2 = 2$ maps are

$$h_1(\boldsymbol{q}) = -3x_1(x_1 x_2 - x_3) + x_2({x_1}^2 + {x_3}^2 + {x_4}^2),$$
$$h_2(\boldsymbol{q}) = -3x_1(x_1 x_3 - x_4) + x_3({x_1}^2 + {x_3}^2 + {x_4}^2).$$

Keeping in mind the (much simpler) structure of the outputs h_k in the $n = 3$ dimensional system, one could try to design a control law by exploiting the existing lower dimensional solution. To this end, note that for $x_4 = 0$, the map $h_1(\boldsymbol{q})$ coincides with the one in the $n = 3$ case, while the second map reads $h_2(\boldsymbol{q}) = -x_3(2{x_1}^2 - {x_3}^2)$; then, $x_3 \to 0$ implies that $h_2 \to 0$, while the convergence of x_3 to zero is already guaranteed for $n = 3$. Consequently, one could resort to finding a way to force $x_4 \to 0$, while keeping the structure of the control solution for the $n = 3$ case.

For the convergence of x_4 to zero, one can require that $\dot{x}_4 = -k_4 x_4,\ k_4 > 0$. Substituting the system equation yields $x_3 u_1 = -k_4 x_4$, which for the control input $u_1 = -k_1 x_1$ reads $k_1 x_3 x_1 = k_4 x_4$. Taking $x_1 \neq 0$, for the same reasons as in $n = 3$ case, one gets $k_1\frac{x_3}{x_1} = k_4\frac{x_4}{{x_1}^2}$; if this condition holds, then $x_4 \to 0$. Consequently, one has to control so that the error $s = k_1\frac{x_3}{x_1} - k_4\frac{x_4}{{x_1}^2}$ converges to zero, while x_1 converges slowly to zero using $u_1 = -k_1 x_1$. In order to use the same control architecture as in the 3-dimensional case, one can take $u_2\big|_{n=4} = u_2\big|_{n=3} + s \Rightarrow$

$$u_2 = -k_2 x_2 + (k_2 + k_1)\frac{x_3}{x_1} + k_1\frac{x_3}{x_1} - k_4\frac{x_4}{{x_1}^2} \Rightarrow$$
$$u_2 = -k_2 x_2 + k_3\frac{x_3}{x_1} - k_4\frac{x_4}{{x_1}^2},\ \text{for } x_1 \neq 0, \tag{23}$$

where $k_3, k_4 > k_2$. In order to study the convergence of the overall system to the origin one can employ a singular perturbation argument, and think of the system as decomposed into two subsystems with different time scales, where the states $\boldsymbol{z} \triangleq \begin{bmatrix} x_2 & x_3 & x_4 \end{bmatrix}^{\mathrm{T}}$ constitute the boundary-layer (fast) system, and the state $\boldsymbol{x} \triangleq x_1$ constitutes the reduced (slow) system. Then, the closed-loop dynamics of the overall system can be written as a singular perturbation model by considering the (small) parameter $\varepsilon = \frac{1}{k_4}$ as

$$\varepsilon\dot{x}_2 = -(1 - a_1\varepsilon)x_2 + (1 - a_2\varepsilon)\frac{x_3}{x_1} - \frac{x_4}{{x_1}^2},$$
$$\varepsilon\dot{x}_3 = -\varepsilon k_1 x_1 x_2,\ \ \varepsilon\dot{x}_4 = -\varepsilon k_1 x_1 x_3,\ \ \dot{x}_1 = -k_1 x_1,$$

where $k_2 = k_4 - a_1,\ k_3 = k_4 - a_2$. The boundary-layer system has one isolated root, given for $\varepsilon = 0$ as $x_2 = \frac{x_3}{x_1} - \frac{x_4}{{x_1}^2}$. Taking $y = x_2 - (\frac{x_3}{x_1} - \frac{x_4}{{x_1}^2})$, one can easily verify that $\frac{dy}{d\tau} = -y$, which implies that x_2 converges exponentially and at a very fast time scale to $\frac{x_3}{x_1} - \frac{x_4}{{x_1}^2}$. The remaining dynamics of the boundary layer system then read $\dot{x}_3 = -k_1 x_1 x_2 = -k_1 x_1(\frac{x_3}{x_1} - \frac{x_4}{{x_1}^2}) = -k_1 x_3 + k_1\frac{x_4}{x_1}$, $\dot{x}_4 = -k_1 x_1 x_3$; for $x_1 \neq 0$ a "frozen" parameter, the resulting linear subsystem of $x_3,\ x_4$ has eigenvalues of negative real part, which are independent of x_1. Thus, with an appropriate choice of the control gains where k_4 sufficiently large so that $\varepsilon \to 0$, one has that the boundary layer system states converge to zero. To tune the control gains, one can write the dynamics of the

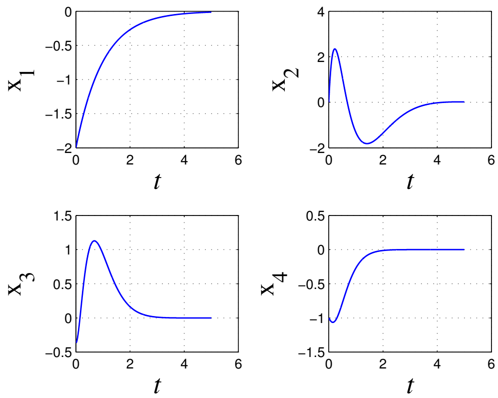


Fig. 4. The state trajectories of a 4-d chained system for $k_1 = 1$, $k_2 = 10$, $k_3 = 42$, $k_4 = 75$.

boundary-layer states in matrix form

$$\begin{bmatrix} \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -k_2 & \frac{k_3}{x_1} & -\frac{k_4}{{x_1}^2} \\ -k_1 x_1 & 0 & 0 \\ 0 & -k_1 x_1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \\ x_4 \end{bmatrix} \Rightarrow \dot{\boldsymbol{z}} = \boldsymbol{A_1}(x_1)\boldsymbol{z},$$

and choose k_1, k_2, k_3, k_4 so that $\boldsymbol{A_1}(x_1)$ is a Hurwitz matrix, and its eigenvalues are small compared to the eigenvalue of the slow subsystem. The closed-loop system trajectories are shown in Fig. 4. Note that the same procedure applies for $\forall\, n > 4$ as well.

IV. Conclusions

This paper presented a control design framework for n-dimensional nonholonomic systems, subject to $\kappa \geq 1$ kinematic Pfaffian constraints. An N-dimensional vector field $\mathbf{F}$ of the form (12) serves as reference to the system. The dimension N $\leq n$ of $\mathbf{F}$ depends on the structure of the constraints, and indicates a foliation $\mathcal{F}$ of the configuration space. This foliation, along with aligning the system's vector field with $\mathbf{F}$, indicates the choice of Lyapunov-like functions V. Switching to different controllers occurs only when the initial conditions belong into specific singularity submanifolds $\mathcal{A}$. The unicycle and chained systems for $n = 3, 4$ were considered as illustrative examples, and control laws were constructed following the same guidelines. Future work can be towards the extension of the methodology into nonholonomic systems with dynamic Pfaffian constraints.

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