Addition of Two Angular Momenta in the Matrix Representation

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1 Addition of two angular momenta

Given two angular momentum operators $\mathbf{L}_1$ and $\mathbf{L}_2$, and the basis sets $\{|j_1 m_1\rangle\}$ and $\{|j_2 m_2\rangle\}$ that diagonalize $\mathbf{L}_1^2$ and $\mathbf{L}_2^2$ respectively, we want to find a new basis that diagonalizes $\mathbf{L}^2$, where $\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2$. $\mathbf{L}$ is simply the addition of two angular momenta. However, the notation could be misleading because in general $\mathbf{L}_1$ and $\mathbf{L}_2$ act on different vector spaces[1]. To do the sum, we need to use the vector space spanned by all the possible combinations of $|j_1 m_1\rangle$ and $|j_2 m_2\rangle$, i.e. $\{|j_1 m_1\rangle \otimes |j_2 m_2\rangle \mid \quad \forall |j_1 m_1\rangle, \forall |j_2 m_2\rangle\}$, where the symbol $\otimes$ is called the *tensor product* or *external product* [1]. Generally speaking, if T and S are tensors, then[2]

$$(S \otimes T)^{i_1 \dots i_l \, i_{l+1} \dots i_{l+m}}_{j_1 \dots j_k \, j_{k+1} \dots j_{k+n}} = S^{i_1 \dots i_l}_{j_1 \dots j_k} T^{i_{l+1} \dots i_m}_{j_{k+1} \dots j_n} \tag{1}$$

$\mathbf{L}_1$ and $\mathbf{L}_2$ can be extended to this tensor product space as $\mathbf{L}_1 \otimes \mathbb{1}_2$ and $\mathbb{1}_1 \otimes \mathbf{L}_2$, respectively. It follows that the total angular momentum is [2]

$$\mathbf{L} = \mathbf{L}_1 \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes \mathbf{L}_2 \tag{2}$$

1.1 Tensor product of matrices

A particular useful representation of matrix tensor product is the so-called *Kronecker product*[3]. Given a matrix A of $m \times n$ and a matrix B of $p \times q$, the Kronecker product

[1] In other words, $\mathbf{L}_1$ and $\mathbf{L}_2$ belong to different dual spaces

[2] Sometimes this is denoted as $\mathbf{L} = \mathbf{L}_1 \oplus \mathbf{L}_2$

is a matrix of $mp \times nq$ given by

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix} \tag{3}$$

2 Example: addition of two 1/2 spins

Let's calculate $\mathbf{L}^2 = (\mathbf{L}_1 \oplus \mathbf{L}_2)^2$ for two $j = 1/2$ particles. For each particle, $\mathbf{L}_i^2$ and L_{iz} where $i = 1, 2$ in the $|j\ m\rangle = |1/2\ \pm 1/2\rangle$ basis are

$$\mathbf{L}_i^2 = \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad L_{iz} = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4}$$

Because both operators commute, they share a common basis

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \tag{5}$$

First, we find a basis for the tensor product space

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{6}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \tag{7}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}_1 \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \tag{8}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}_1 \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{9}$$

In this basis, the operator $\mathbf{L}_1$ is

$$\mathbf{L}_1 \otimes \mathbb{1} = \frac{\hbar}{2}(\sigma_x \hat{x} + \sigma_y \hat{y} + \sigma_z \hat{z}) \otimes \mathbb{1} \tag{10}$$

$$= \frac{\hbar}{2}(\sigma_x \otimes \mathbb{1}\hat{x} + \sigma_y \otimes \mathbb{1}\hat{y} + \sigma_z \otimes \mathbb{1}\hat{z}) \tag{11}$$

where σ_i are the Pauli matrices. In the same way, the operator $\mathbf{L}_2$ is

$$\mathbb{1} \otimes \mathbf{L}_2 = \frac{\hbar}{2}\, \mathbb{1} \otimes (\sigma_x \hat{x} + \sigma_y \hat{y} + \sigma_z \hat{z}) \tag{12}$$

$$= \frac{\hbar}{2}\, (\mathbb{1} \otimes \sigma_x \hat{x} + \mathbb{1} \otimes \sigma_y \hat{y} + \mathbb{1} \otimes \sigma_z \hat{z}) \tag{13}$$

Therefore, the total angular momentum is[3]

$$\mathbf{L} = \frac{\hbar^2}{4}\, (\sigma_x \otimes \sigma_x \hat{x} + \sigma_y \otimes \sigma_y \hat{y} + \sigma_z \otimes \sigma_z \hat{z}) \tag{14}$$

where

$$\sigma_x \otimes \sigma_x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad \sigma_y \otimes \sigma_y = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{15}$$

$$\sigma_z \otimes \sigma_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{16}$$

Finally

$$\mathbf{L}^2 = \hbar^2 \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \tag{17}$$

[3]The following property of tensor product is used: $(A \otimes B)(C \otimes D) = AC \otimes BC$

2.1 Some properties of $\mathbf{L}^2$

Now let's find the eigenvalue of $\mathbf{L}^2$ and see that we get the *triplets* and *singlet* states. The diagonalization is straightforward, and the eigenvalues are $\{2, 2, 2, 0\}$, with the respective eigenvectors

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right\} \tag{18}$$

The operators $\mathbf{L}^2, L_z, L_1^2, L_2^2$ commute among themselves, but $\mathbf{L}^2$ does not commute with L_{1z} or L_{2z}

References

[1] mathworld.wolfram.com, tensor product

[2] www.wikipedia.com, tensor product

[3] mathworld.wolfram.com, Kronecker product