# Phase-Locked Loop

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I will present a mathematical description of a *phase-locked loop (PLL)*. A PLL is a variable frequency oscillator that matches the phase of an oscillatory reference signal. The output of a PLL has the same oscillation frequency as the input and a constant relative phase. Typical applications are found in synchronous communication between electronic devices, like Ethernet and USB.

The different parts of a PLL are

1. The *reference signal* $r(t)$ is the input of the PPL. It is the signal that we want to phase lock to. For the purpose of our calculation, let us assume it is a sine function $r(t) = A\sin(\omega_0 t)$

2. The *mixer* multiplies the input and the feedback loop that carries the output of the PLL, resulting in a high and a low frequency component.

3. The *low-pass filter* suppresses the high frequency component.

4. The *voltage-controlled oscillator (VCO)* is an oscillator whose output frequency varies linearly with the input voltage $V$.

5. The *feedback loop* samples the output $y(t)$ and feeds it back to the mixer.

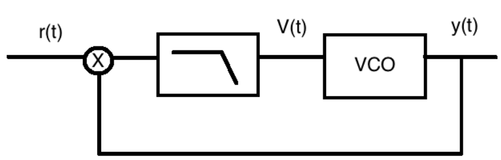


Figure 1: Feedback scheme of a PLL

For concreteness, we will assume that the output of the PLL is a sinusoidal function, i.e.

$$y(t) = B\cos\left(\int_0^t \omega\left(\tau\right)d\tau\right). \tag{1}$$

and that the frequency $\omega(t)$ is the sum of a local oscillator term $\omega_L$ and a term proportional to the input voltage $V(t)$, i.e. $\omega(t) = \omega_L + KV(t)$. Then

$$y(t) = B\cos\left(\omega_L t + \varphi(t)\right) \tag{2}$$

where we defined the time-dependent phase as $\varphi(t) \equiv K\int_0^t V\left(\tau\right)d\tau$

The mixer gives the error signal of the reference and the output of the PLL

$$\begin{aligned} e(t) &= r(t)y(t) \\ &= AB\sin\left(\omega_0 t\right)\cos\left(\omega_L t + \varphi(t)\right) \\ &= \frac{AB}{2}\sin\left(\left(\omega_0 + \omega_L\right)t + \varphi(t)\right) + \frac{AB}{2}\sin\left(\left(\omega_0 - \omega_L\right)t - \varphi(t)\right) \end{aligned} \tag{3}$$

Only the second term on the r.h.s survives the low-pass filter. Let us call this signal $V(t)$

$$V(t) = \frac{AB}{2}\sin\left(\left(\omega_0 - \omega_L\right)t - \varphi(t)\right) \tag{4}$$

By definition

$$\begin{aligned} \frac{d\varphi(t)}{dt} &= KV(t) \\ &= \frac{KAB}{2}\sin\left(\left(\omega_0 - \omega_L\right)t - \varphi(t)\right) \end{aligned} \tag{5}$$

Let us introduce a new variable $\xi(t)$

$$\varphi(t) = \left(\omega_0 - \omega_L\right)t + \xi(t) \tag{6}$$

Then, from Eq.(5)

$$\omega_0 - \omega_L + \frac{d\xi(t)}{dt} = -\frac{KAB}{2}\sin\xi(t) \tag{7}$$

We can linearize Eq.(7) [1]

$$\omega_0 - \omega_L + \frac{d\xi(t)}{dt} = -\frac{KAB}{2}\xi(t) \tag{8}$$

The solution of the differential equation is

$$\xi(t) = \xi_0 e^{-t/t_0} - (\omega_0 - \omega_L)\, t_0 \tag{9}$$

where $t_0 \equiv 2(KAB)^{-1}$. Replacing Eq.(9) in Eq.(6)

$$\varphi(t) = (\omega_0 - \omega_L)\, t + \xi_0 e^{-t/t_0} - (\omega_0 - \omega_L)\, t_0 \tag{10}$$

Under steady state condition $(t \gg t_0)$

$$\boxed{\varphi(t) = (\omega_0 - \omega_L)\, t - (\omega_0 - \omega_L)\, t_0} \tag{11}$$

We can see that in fact the output $y(t)$ phase locks to the input

$$\begin{aligned} y(t) &= B\cos(\omega_L t + \varphi(t)) \\ &= B\cos(\omega_L t + (\omega_0 - \omega_L)\, t - (\omega_0 - \omega_L)\, t_0) \\ &= B\cos(\omega_0 t - (\omega_0 - \omega_L)\, t_0) \end{aligned} \tag{12}$$

In conclusion, the output $y(t)$ is in sync with the reference $r(t)$.

[1]It would be interesting to study the range of stability for this particular PLL.