Quantum Free Particle Time Evolution

David Chen

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The time-dependent Schrödinger equation is

$$\hat{\mathcal{H}}(t) \left|\psi(t)\right\rangle = i\hbar \frac{\partial}{\partial t} \left|\psi(t)\right\rangle \tag{1}$$

For a time-independent Hamiltonian $\hat{\mathcal{H}}$, the time evolution of the wave-function is

$$\left|\psi(t)\right\rangle = e^{-i\frac{\hat{\mathcal{H}}}{\hbar}t} \left|\psi_0\right\rangle \tag{2}$$

Projecting it onto $\langle \mathbf{r}|$

$$\begin{aligned} \langle \mathbf{r}|\psi(t)\rangle &= \langle \mathbf{r}| \, e^{-i\frac{\hat{\mathcal{H}}}{\hbar}t} \left|\psi_0\right\rangle & (3) \\ &= \int d^n r' \, \underbrace{\langle \mathbf{r}| \, e^{-i\frac{\hat{\mathcal{H}}}{\hbar}t} \left|\mathbf{r}'\right\rangle}_{\equiv \mathcal{K}(\mathbf{r},\mathbf{r}';t)} \langle \mathbf{r}'|\psi_0\rangle & (4) \end{aligned}$$

where $\mathcal{K}(\mathbf{r}, \mathbf{r}'; t)$ is the *quantum propagator* and n is the dimensionality of space.

In free space, $\hat{\mathcal{H}} = \hat{\mathbf{p}}^2/2m$, therefore [1]

$$\begin{aligned} \mathcal{K}(\mathbf{r}, \mathbf{r}'; t) &= \langle \mathbf{r}| \, e^{-i\frac{\hat{\mathbf{p}}^2 t}{2m\hbar}} \left|\mathbf{r}'\right\rangle & (5) \\ &= \int \langle \mathbf{r}|\mathbf{p}'\rangle \, d^n p' \, \langle \mathbf{p}'| \, e^{-i\frac{\hat{\mathbf{p}}^2 t}{2m\hbar}} \left|\mathbf{p}\right\rangle d^n p \, \langle \mathbf{p}'|\mathbf{r}\rangle & (6) \\ &= \frac{1}{(2\pi\hbar)^n} \int d^n p \, d^n p' \, e^{i\frac{\mathbf{r}\cdot\mathbf{p}'}{\hbar}} e^{-i\frac{\mathbf{p}^2 t}{2m\hbar}} \, \delta(\mathbf{p}' - \mathbf{p}) \, e^{-i\frac{\mathbf{r}'\cdot\mathbf{p}}{\hbar}} & (7) \\ &= \frac{1}{(2\pi\hbar)^n} \int d^n p \, e^{-i\frac{\mathbf{p}^2 t}{2m\hbar}} \, e^{i\frac{\mathbf{p}}{\hbar}\cdot(\mathbf{r}-\mathbf{r}')} & (8) \end{aligned}$$

[1] $\langle \mathbf{p}'|\mathbf{r}\rangle = (2\pi\hbar)^{-n/2} \, e^{-i\frac{\mathbf{r}'\cdot\mathbf{p}}{\hbar}}$; $\langle \mathbf{p}'| \, e^{-i\frac{\hat{\mathbf{p}}^2 t}{2m\hbar}} \left|\mathbf{p}\right\rangle = e^{-i\frac{\mathbf{p}^2 t}{2m\hbar}} \langle \mathbf{p}'|\mathbf{p}\rangle = e^{-i\frac{\mathbf{p}^2 t}{2m\hbar}} \delta\left(\mathbf{p}' - \mathbf{p}\right)$

Completing the square of the exponent

$$-\frac{it}{2m\hbar}\left(\mathbf{p}^2-\frac{2m}{t}(\mathbf{r}-\mathbf{r}')\cdot\mathbf{p}\right)=-\frac{it}{2m\hbar}\left(\mathbf{p}-\frac{m}{t}(\mathbf{r}-\mathbf{r}')\right)^2+\frac{im}{2\hbar t}(\mathbf{r}-\mathbf{r}')^2 \tag{9}$$

Therefore

$$\mathcal{K}(\mathbf{r},\mathbf{r}';t)=\frac{1}{(2\pi\hbar)^n}e^{\frac{im}{2\hbar t}(\mathbf{r}-\mathbf{r}')^2}\underbrace{\int d^np\; e^{-\frac{it}{2m\hbar}\left(\mathbf{p}-\frac{m}{t}(\mathbf{r}-\mathbf{r}')\right)^2}}_{\left(\frac{2\pi m\hbar}{it}\right)^{n/2}} \tag{10}$$

$$=\left(\frac{m}{2\pi i\hbar t}\right)^{n/2}e^{\frac{im}{2\hbar t}(\mathbf{r}-\mathbf{r}')^2} \tag{11}$$

The following well know property was used in Eq.(10): $\int_{-\infty}^{\infty}e^{-\alpha p^2}=\sqrt{\frac{\pi}{\alpha}}\quad\forall\alpha\in\mathbb{C},\ \Re\{\alpha\}>0$. The exponent of the integrand in Eq.(10) is a pure imaginary number. Strictly speaking, we can not use the property above. However, we can add a positive real number $\epsilon>0$ in the exponent and then $\epsilon\to0$ after integrating.

Finally

$$\boxed{\psi(\mathbf{r},t)=\left(\frac{m}{2\pi i\hbar t}\right)^{n/2}\int d^nr'\; e^{\frac{im}{2\hbar t}(\mathbf{r}-\mathbf{r}')^2}\psi_0\left(\mathbf{r}'\right)} \tag{12}$$

References

[1] Shankar, R. Principles of Quantum Mechanics, 2nd ed., Chapter 5.

[2] Pethick, C. J., Bose-Einstein Condensation in Dilute Gases, 2nd ed., Chapter 2