Matrix Representation of Angular Momentum

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1 Angular Momentum

In Quantum Mechanics, the angular momentum operator $\mathbf{L} = \mathbf{r} \times \mathbf{p} = L_x \hat{x} + L_y \hat{y} + L_z \hat{z}$ satisfies

$$\mathbf{L}^2 \left| jm \right\rangle = \hbar \, j(j+1) \left| jm \right\rangle \tag{1}$$

$$L_z \left| jm \right\rangle = \hbar \, m \left| jm \right\rangle \tag{2}$$

The demonstration can be found in any Quantum Mechanics book, and it follows from the commutation relation $[\mathbf{r}, \mathbf{p}] = i\hbar\mathbb{1}$

It is useful to define the rising and lowering operators $L_{\pm} \equiv L_x \pm i L_y$, which have the following property

$$L_{\pm} \left| jm \right\rangle = \hbar \, \sqrt{j(j+1) - m(m \pm 1)} \left| jm \pm 1 \right\rangle \tag{3}$$

And L_x and L_y are obtained from

$$\begin{aligned} L_x &= (L_+ + L_-)/2 \\ L_y &= (L_+ - L_-)/2i \end{aligned} \tag{4}$$

1.1 Spin 1/2

If $j = 1/2$, the spin-space is spanned by two states: $\{\left|1/2\ 1/2\right\rangle, \left|1/2\ \text{-}1/2\right\rangle\}$. The properties Eq.(2) and Eq.(3) for this particular case are

$$L_z \left|1/2\ \pm 1/2\right\rangle = \pm\hbar/2 \left|1/2\ \pm 1/2\right\rangle \tag{5}$$

$$L_+ \left|1/2\ 1/2\right\rangle = 0 \tag{6}$$

$$L_+ \left|1/2\ \text{-}1/2\right\rangle = \hbar \left|1/2\ 1/2\right\rangle \tag{7}$$

$$\tag{8}$$

If we use the matrix representation $(1\ 0)^T \equiv |1/2\ 1/2\rangle$ and $(0\ 1)^T \equiv |1/2\ \text{-}1/2\rangle$, the operators are

$$L_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \quad L_+ = \hbar\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \quad L_- = L_+^\dagger \tag{9}$$

and from Eqs.(4)

$$L_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} \quad L_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \tag{10}$$

and $\mathbf{L} = \hbar/2\,(\sigma_x\hat{x} + \sigma_y\hat{y} + \sigma_z\hat{z})$, where σ_i are the Pauli matrices. $\mathbf{L}^2$ and L_z are both diagonal in this basis set, as expected from Eq.(1) and Eq.(2)

$$\mathbf{L}^2 = \frac{3}{4}\hbar^2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \quad L_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{11}$$

1.2 Spin 3/2

In this case, the spin-space is spanned by four states: $\{|3/2\ \pm 3/2\rangle\,, |3/2\ \pm 1/2\rangle\}$. If we choose the following matrix representation

$$(1\quad 0\quad 0\quad 0)^T \equiv |3/2\ 3/2\rangle \tag{12}$$
$$(0\quad 1\quad 0\quad 0)^T \equiv |3/2\ 1/2\rangle \tag{13}$$
$$(0\quad 0\quad 1\quad 0)^T \equiv |3/2\ \text{-}1/2\rangle \tag{14}$$
$$(0\quad 0\quad 0\quad 1)^T \equiv |3/2\ \text{-}1/2\rangle \tag{15}$$

and following the same procedure as before gives

$$L_z = \frac{\hbar}{2}\begin{pmatrix}3 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -3\end{pmatrix} \quad L_+ = \hbar\begin{pmatrix}0 & \sqrt{3} & 0 & 0\\ 0 & 0 & \sqrt{4} & 0\\ 0 & 0 & 0 & \sqrt{3}\\ 0 & 0 & 0 & 0\end{pmatrix} \quad L_- = L_+^\dagger \tag{16}$$

$$L_x = \frac{\hbar}{2}\begin{pmatrix}0 & \sqrt{3} & 0 & 0\\ \sqrt{3} & 0 & \sqrt{4} & 0\\ 0 & \sqrt{4} & 0 & \sqrt{3}\\ 0 & 0 & \sqrt{3} & 0\end{pmatrix} \quad L_y = \frac{i\hbar}{2}\begin{pmatrix}0 & \text{-}\sqrt{3} & 0 & 0\\ \sqrt{3} & 0 & \text{-}\sqrt{4} & 0\\ 0 & \sqrt{4} & 0 & \text{-}\sqrt{3}\\ 0 & 0 & \sqrt{3} & 0\end{pmatrix} \tag{17}$$

and the total angular momentum satisfies $\mathbf{L}^2 = 15\hbar^2/4 \cdot \mathbb{1}$

References

[1] Gasiorowics, Quantum Physics