Commutative Operators and Common Basis

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1 Non-degenerate case

Given two operators A and B that satisfy the commutation relation, i.e. $[A, B] = AB - BA = 0$, and given a non-degenerate eigenstate of A, $|a\rangle$, we have the following relation

$$A\left(B\left|a\right\rangle\right) = BA\left|a\right\rangle \tag{1}$$
$$= Ba\left|a\right\rangle \tag{2}$$
$$= a\left(B\left|a\right\rangle\right) \tag{3}$$

In other words, $B|a\rangle$ is an eigenstate of A with eigenvalue a. Since $|a\rangle$ is non-degenerate, $B|a\rangle$ has to be proportional to $|a\rangle$, or

$$B\left|a\right\rangle = b\left|a\right\rangle \tag{4}$$

Therefore, $|a\rangle$ is simultaneously an eigenstate of A and B, and it is usually labeled by their eigenvalues, i.e. $|a, b\rangle$

2 Degenerate case

For simplicity, let's assume a twofold degeneracy, but it can be easily generalized.

$$A\left|a^{(1)}\right\rangle = a\left|a^{(1)}\right\rangle \tag{5}$$
$$A\left|a^{(2)}\right\rangle = a\left|a^{(2)}\right\rangle \tag{6}$$

In the way as above, $B\left|a^{(1)}\right\rangle$ and $B\left|a^{(2)}\right\rangle$ are eigenstates of A with eigenvalue a. However, because $\left|a^{(1)}\right\rangle$ and $\left|a^{(2)}\right\rangle$ are degenerate, we can only state that

$$B\left|a^{(1)}\right\rangle = b_{11}\left|a^{(1)}\right\rangle + b_{12}\left|a^{(2)}\right\rangle \tag{7}$$

$$B\left|a^{(2)}\right\rangle = b_{21}\left|a^{(1)}\right\rangle + b_{22}\left|a^{(2)}\right\rangle \tag{8}$$

We can diagonalize B in the sub-space associated to a, which is spanned by $\left\{\left|a^{(1)}\right\rangle, \left|a^{(2)}\right\rangle\right\}$, resulting [1]

$$\begin{aligned} B\left|b^{(1)}\right\rangle &= b_1\left|b^{(1)}\right\rangle \\ B\left|b^{(2)}\right\rangle &= b_2\left|b^{(2)}\right\rangle \end{aligned} \tag{9}$$

Finally, we can express $\left|b^{(i)}\right\rangle$ in terms of $\left\{\left|a^{(1)}\right\rangle\right\}$ and $\left\{\left|a^{(2)}\right\rangle\right\}$

$$\left|b^{(i)}\right\rangle = \sum_{j=1,2} \left\langle a^{(j)} \middle| b^{(i)}\right\rangle \left|a^{(j)}\right\rangle \tag{10}$$

$\left|b^{(1)}\right\rangle$ and $\left|b^{(2)}\right\rangle$ are eigenstates of B by definition, Eq.(9), and we can see from Eq.(10) that they are also eigenstates of A. As before, the state $\left|b^{(i)}\right\rangle$ is usually denoted as $|a, b\rangle$

3 Example

The Hamiltonian operator of a free particle is

$$\hat{\mathcal{H}} = \frac{\hat{p}^2}{2m} \tag{11}$$

There are two degenerate eigenstates for this operator, and the common eigenvalue is $p^2/2m$. On the other hand, it is clear that the momentum operator $\hat{p}$ commutes with $\hat{\mathcal{H}}$. Therefore, there is a common basis for both operators, which is $\{|E, p\rangle, |E, -p\rangle\}$ where $E = p^2/2m$.

References

[1] Gasiorowics, Quantum Physics

[2] Sakurai, Modern Quantum Mechanics

[1] Assuming that $\left|a^{(1)}\right\rangle$ and $\left|a^{(2)}\right\rangle$ are orthogonal. What happens in the general case?