Non-interacting Bose-Einstein Condensate

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The partition function in the grand canonical ensemble is

$$\Xi = \sum_{l} e^{-\beta(E_l - \mu N_l)} \tag{1}$$

In the case of non-interacting and indistinguishable particles

$$E_l = N_1\epsilon_1 + N_2\epsilon_2 + \ldots$$
$$N = N_1 + N_2 + \ldots$$

Therefore

$$\begin{aligned}
\Xi &= \sum_{N_1, N_2, \ldots} e^{-\beta(N_1\epsilon_1 + N_2\epsilon_2 + \ldots - \mu(N_1 + N_2 + \ldots))} \\
&= \sum_{N_1} e^{-\beta N_1(\epsilon_1 - \mu)} \cdot \sum_{N_2} e^{-\beta N_2(\epsilon_2 - \mu)} \cdot \ldots \\
&= \prod_{\lambda=1}^{\infty} \sum_{N_\lambda} e^{-\beta N_\lambda(\epsilon_\lambda - \mu)} \\
&= \prod_{\lambda=1}^{\infty} \xi_\lambda
\end{aligned} \tag{2}$$

where $\xi_\lambda \equiv \sum_{N_\lambda} e^{-\beta N_\lambda(\epsilon_\lambda - \mu)}$ is the partition function of the state λ.

Given that the probability of occupying the state l is $P_l \equiv e^{-\beta(E_l - \mu N_l)}/\Xi$, the mean occupation number is

$$
\begin{aligned}
\bar{N} &= \frac{1}{\Xi}\sum_{l} N_l e^{-\beta(E_l-\mu N_l)} \\
&= \frac{1}{\beta\Xi}\sum_{l} \frac{\partial}{\partial\mu} e^{-\beta(E_l-\mu N_l)} \\
&= \frac{1}{\beta\Xi}\frac{\partial}{\partial\mu}\sum_{l} e^{-\beta(E_l-\mu N_l)} \\
&= \frac{1}{\beta}\frac{\partial}{\partial\mu}\ln\Xi \\
&= \sum_{\lambda=1}^{\infty}\frac{1}{\beta}\frac{\partial}{\partial\mu}\ln\xi_\lambda \\
&= \sum_{\lambda=1}^{\infty}\bar{N}_\lambda
\end{aligned}
\tag{3}
$$

where $\bar{N}_\lambda \equiv \frac{1}{\beta}\frac{\partial}{\partial\mu}\ln\xi_\lambda$

The equations derived so far are equally valid for fermions or bosons. For the particular case of bosons we have $\xi_\lambda = \sum_{N_\lambda=0}^{\infty}\left(e^{-\beta(\epsilon_\lambda-\mu)}\right)^{N_\lambda} = \frac{1}{1-e^{-\beta(\epsilon_\lambda-\mu)}}$ and, therefore, $\bar{N}_\lambda = \frac{1}{e^{\beta(\epsilon_\lambda-\mu)}-1}$. Hence, the mean occupation number is

$$
\bar{N} = \sum_{\lambda}\frac{1}{e^{\beta(\epsilon_\lambda-\mu)}-1}
\tag{4}
$$

It is more convenient to sum over energies rather than individual states λ. If we use the density of states g, which has the general form $g(\epsilon) = C_\alpha \epsilon^{\alpha-1}$, then

$$
\begin{aligned}
\bar{N} &= \sum_{\epsilon}\frac{g(\epsilon)}{e^{\beta(\epsilon_\lambda-\mu)}-1} \\
&= \int d\epsilon \frac{g(\epsilon)}{e^{\beta(\epsilon_\lambda-\mu)}-1} \\
&= C_\alpha \int d\epsilon \frac{\epsilon^{\alpha-1}}{e^{\beta(\epsilon_\lambda-\mu)}-1}
\end{aligned}
\tag{5}
$$

The parameter α depends on the spatial configuration of the trap. For example, in free space $\alpha = 3/2$ and in a quadratic trap $\alpha = 3$.

The condensation temperature T_c is defined as the highest temperature at which the lowest-energy state becomes macroscopically occupied.

Right on T_c, the chemical potential μ is zero (why?), and the eq.(5) becomes

$$\bar{N} = C_\alpha \Gamma(\alpha) \zeta(\alpha) (kT_c)^\alpha \tag{6}$$

At $T < T_c$, the presence of a condensed phase implies that $\bar{N} = \bar{N}_0 + \bar{N}_{ex}$, where $\bar{N}_{ex}$ is the number of excited atoms

$$\begin{aligned} \bar{N}_{ex} &\equiv \int d\epsilon C_\alpha \frac{\epsilon^{\alpha-1}}{e^{\beta(\epsilon_\lambda - \mu)} - 1} \\ &= C_\alpha \Gamma(\alpha) \zeta(\alpha) (kT)^\alpha \end{aligned} \tag{7}$$

Therefore, combining (6) and (7)

$$\frac{\bar{N}_{ex}}{\bar{N}} = \left(\frac{T}{T_c}\right)^\alpha \quad \text{and} \quad \frac{\bar{N}_0}{\bar{N}} = 1 - \left(\frac{T}{T_c}\right)^\alpha \tag{8}$$

The phase-space density is defined as $\varpi \equiv \bar{n}\lambda_T^3$, where $\bar{n} \equiv \bar{N}/V$ is the particle density and $\lambda_T = \frac{h}{\sqrt{mk_BT}}$ is the de Broglie wavelength.

Note: we have kept the mean value symbol on N to highlight the fact that in the grand canonical ensemble the atom number is not fixed, but fluctuates around the mean.

References

[1] C. J. Pethick, "Bose-Einstein Condensation in Dilute Gases"