Anti-Helmholtz magnetic trap

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1 Electromagnetism review

A point charge q located at $\mathbf{r}_q$ and moving at a velocity $\mathbf{v}_q$ generates the magnetic field

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} q\,\mathbf{v_q} \times \frac{\mathbf{r}-\mathbf{r}_q}{|\mathbf{r}-\mathbf{r}_q|^3}. \tag{1}$$

In the case of a charge distribution, we replace $q\,\mathbf{v}_q$ by $d^3r\,\mathbf{J}$, where $\mathbf{J}$ is the current flow. The magnetic field becomes

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r'\,\mathbf{J}(\mathbf{r}') \times \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \tag{2}$$

For a unidimensional conductor we can integrate the transverse direction. Equation (2) reduces to the Biot-Savart law,

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \int d\mathbf{r}' \times \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}, \tag{3}$$

where $d\mathbf{r}'$ is a line integration and I is the current through the wire.

Far from the conductor, i.e. $|\mathbf{r}| \ll |\mathbf{r}'|$, the eq. (3) can be expanded via the series $\frac{1}{|\mathbf{r}-\mathbf{r}'|^3} = \frac{1}{|\mathbf{r}'|^3} + \frac{3\,\mathbf{r}\cdot\mathbf{r}'}{|\mathbf{r}'|^5} + \ldots$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \int d\mathbf{r}' \times (\mathbf{r}-\mathbf{r}')\frac{1}{|\mathbf{r}'|^3} + \frac{3\mu_0 I}{4\pi} \int d\mathbf{r}' \times (\mathbf{r}-\mathbf{r}')\frac{\mathbf{r}\cdot\mathbf{r}'}{|\mathbf{r}'|^5}. \tag{4}$$

2 Anti-Helmholtz trap

This quadrupole trap is formed by two identical ring-shape wires shifted vertically and carrying current in opposite directions.

Let us first calculate the expression in (4) for a single ring of radius R and shifted vertically in $+a$. The integration coordinate is $\mathbf{r}' = R\cos\theta'\hat{x} + R\sin\theta'\hat{y} + a\hat{z}$ and its differential is $d\mathbf{r}' = R\hat{\theta}d\theta'$. Let us consider

$$\begin{aligned}\hat{\theta} \times (\mathbf{r} - \mathbf{r}') &= (-\sin\theta', \cos\theta', 0) \times (x - R\cos\theta', y - R\sin\theta', -a) \\ &= (-a\cos\theta', -a\sin\theta', R - y\sin\theta' - x\cos\theta')\end{aligned} \tag{5}$$

and

$$\mathbf{r} \cdot \mathbf{r}' = xR\cos\theta' + yR\sin\theta' + za. \tag{6}$$

Given eqs. (5) and (6), the first and second integrals on the RHS of (4) are $\frac{\mu_0 I R^2}{2(R^2+a^2)^{3/2}}\hat{z}$ and $\frac{3\mu_0 I a R^2}{2(R^2+a^2)^{5/2}}(-\frac{x}{2}, -\frac{y}{2}, z)$ respectively (only the linear terms in $\mathbf{r}$ were kept). Therefore, the field is

$$\mathbf{B}_{+a}(\mathbf{r}) = \frac{\mu_0 I R^2}{2(R^2+a^2)^{3/2}}\hat{z} + \frac{3\mu_0 I a R^2}{2(R^2+a^2)^{5/2}}\left(-\frac{x}{2}, -\frac{y}{2}, z\right). \tag{7}$$

To calculate the field produced by the second ring, we can simply do $I \to -I$ and $a \to -a$. This leads to

$$\mathbf{B}_{-a}(\mathbf{r}) = -\frac{\mu_0 I R^2}{2(R^2+a^2)^{3/2}}\hat{z} + \frac{3\mu_0 I a R^2}{2(R^2+a^2)^{5/2}}\left(-\frac{x}{2}, -\frac{y}{2}, z\right). \tag{8}$$

Finally, the total field is[1]

$$\mathbf{B}(\mathbf{r}) \equiv \mathbf{B}_{+a}(\mathbf{r}) + \mathbf{B}_{-a}(\mathbf{r}) = B_0\left(\frac{x}{2}, \frac{y}{2}, -z\right), \tag{9}$$

where $B_0 \equiv -\frac{3\mu_0 I a R^2}{2(R^2+a^2)^{5/2}}$; and the potential is

$$U_{mag}(\mathbf{r}) = g_F m_F \mu_B |\mathbf{B}(\mathbf{r})|. \tag{10}$$

[1] I think these results are calculated in Stan's thesis via a multipole expansion.