ECE 310 : Lecture : FREQUENCY RESPONSE

$$x[n] \rightarrow \boxed{\text{LTI}} \rightarrow y[n]$$

$$y[n] = \sum_{m=-\infty}^{\infty} h[m]\, x[n-m]$$

$$\boxed{\sum_{n=-\infty}^{\infty} |h[n]| < \infty}$$

By Convolution property :

$$\boxed{Y_d(\omega) = H_d(\omega)\, X_d(\omega)}$$

Here $H_d(\omega)$ is the DTFT of $h[n]$ (impulse Response)

$\boxed{H_d(\omega) \rightarrow \text{Frequency Response of the System}}$

Frequency Response :

① Completely characterises the system in the frequency domain.

② Defines the relationship in the frequency domain between $X_d(\omega)$ and $Y_d(\omega)$.

RESPONSE TO COMPLEX EXPONENTIAL AND SINUSOIDAL SIGNALS

Let us excite an LTI system with a complex exponential input in order to get a frequency domain characterization of the system.

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$$x[n] = A e^{j\omega_0 n} \qquad -\infty \leq n \leq \infty \;, \; -\pi \leq \omega_0 \leq \pi$$

i.e $\boxed{x[n] = \text{Complex exponential at frequency } \omega = \omega_0}$

$$y[n] = \sum_{m=-\infty}^{\infty} h[m]\, x[n-m]$$

$$\Rightarrow \quad y[n] = \sum_{m=-\infty}^{\infty} h[m]\, A e^{j\omega(n-m)}$$

$$= \sum_{m=-\infty}^{\infty} h[m]\, e^{j\omega_0 n} \cdot e^{-j\omega_0 m}$$

$$\Rightarrow \quad y[n] = A e^{-j\omega_0 n} \cdot \sum_{m=-\infty}^{\infty} h[m]\, e^{-j\omega_0 m}$$

$$\Rightarrow \boxed{y[n] = A e^{j\omega_0 n} H_d(\omega_0)}$$

Note the output in time-domain is the input scaled by a multiplicate factor ($H_d(\omega_0)$)

$H_d(\omega_0)$ = Frequency Response $H_d(\omega)$ evaluated at $\boxed{\omega = \omega_0}$

i.e $\boxed{H_d(\omega_0) = H_d(\omega)\big|_{\omega=\omega_0}}$

$x[n] = A e^{j\omega_0 n}$ is also called the eigen function of the system.

Eigen-function: An input signal that produces an output that differs from the input by constant multiplicative factor. The multiplicative factor is called "eigen-value" of the system.

Also note:

$$y[n] = A e^{j\omega_0 n} \cdot H_d(\omega_0)$$

$$= A e^{j\omega_0 n} |H_d(\omega_0)| e^{j \angle H_d(\omega_0)}$$

$$\Rightarrow \boxed{y[n] = |H_d(\omega_0)| A \, e^{j(\omega_0 n + \angle H_d(\omega_0))}}$$

magnitude scaled by magnitude Response at frequency ω_0

phase changes by phase Response

Example:

① $y[n] = x[n] + 2x[n-1]$

Find output for $x[n] = j^n$

$x[n] = j^n \Rightarrow \boxed{x[n] = e^{j\frac{\pi}{2}n}}$

$$Y(z) = X(z) + 2X(z) \cdot z^{-1}$$

$$\Rightarrow H(z) = \frac{Y(z)}{X(z)} = 1 + 2z^{-1}$$ ROC: z-plane except $z = 0$

$$H_d(\omega) = 1 + 2e^{-j\omega}$$

$$\Rightarrow \quad H_d(\omega)\Big|_{\omega=\frac{\pi}{2}} = 1 + 2e^{-j\pi/2}$$

$$\Rightarrow \boxed{H_d\left(\tfrac{\pi}{2}\right) = 1 - 2j}$$

$$y[n] = e^{j\frac{\pi}{2}n} \cdot H_d\left(\tfrac{\pi}{2}\right)$$

$$\Rightarrow \boxed{y[n] = e^{j\frac{\pi}{2}n}(1-2j)}$$

RESPONSE TO SINUSOIDS:



$$\cos\omega_0 n = \frac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2}$$

$$\Rightarrow \quad y[n] = \frac{1}{2} e^{j\omega_0 n} H_d(\omega_0) + \frac{1}{2} H_d(-\omega_0) \cdot e^{-j\omega_0 n}$$

$\cos\omega_0 n$ is <u>not</u> an eigen function of the LTI system

Assume $h[n]$ = Real-valued.

$$\Rightarrow \quad y[n] = \frac{1}{2} e^{j\omega_0 n} H_d(\omega_0) + \frac{1}{2} e^{-j\omega_0 n} \cdot H_d^*(\omega_0)$$

$$= \frac{1}{2} e^{j\omega_0 n} |H_d(\omega_0)| e^{j\angle H_d(\omega_0)} + \frac{1}{2} e^{-j\omega_0 n} |H_d(\omega_0)| e^{-j\angle H_d(\omega_0)}$$

$$= \frac{1}{2} |H_d(\omega_0)| e^{j(\omega_0 n + \angle H_d(\omega_0))} + \frac{1}{2} |H_d(\omega_0)| e^{-j(\omega_0 n + \angle H_d(\omega_0))}$$

$$\Rightarrow \quad \boxed{y[n] = |H_d(\omega_0)| \cos(\omega_0 n + \angle H_d(\omega_0))} \rightarrow \text{for real-valued } h[n]$$

The output is cosine at 'same frequency'. But $\cos\omega_0 n$ is <u>not</u> an eigen function.

Similarly for $x[n] = \sin \omega_0 n$

$$\Rightarrow \boxed{y[n] = |H_d(\omega_0)| \sin(\omega_0 n + \angle H_d(\omega_0))}$$

① $|H_d(\omega)|$ and $\angle H_d(\omega)$ completely characterize the effect of the system on a sinusoidal input signal of any arbitrary frequency.

② $|H_d(\omega)|$ determines amplification or attenuation imparted by the system and $\angle H_d(\omega)$ determines the phase shift.

Examples:

① $y[n] = x[n] + x[n-1]$ $\quad x[n] = 1, \cos\frac{\pi}{2}n, (-1)^n$

$$\Rightarrow Y(z) = X(z) + z^{-1} X(z)$$

$$H(z) = 1 + z^{-1} \quad \text{ROC includes unit circle.}$$

$$\Rightarrow \boxed{H_d(\omega) = 1 + e^{-j\omega}}$$

a) $x[n] = 1 = e^{j0n} \Rightarrow \omega_0 = 0$

$$\therefore H_d(0) = 1 + e^{-j0} \Rightarrow \boxed{H_d(0) = 2}$$

$$|H_d(0)| = 2 \qquad \angle H_d(0) = 0^\circ$$

$$\Rightarrow y[n] = |H_d(0)| e^{j \cdot 0 n} \cdot e^{j \angle H_d(0)}$$

$$\Rightarrow \boxed{y[n] = 2}$$

(b) $x[n] = \cos\frac{\pi}{2}n$ $\quad$ Note $h[n]$ = Real-valued

$$\Rightarrow y[n] = |H_d(\tfrac{\pi}{2})| \cos(\tfrac{\pi}{2}n + \angle H_d(\tfrac{\pi}{2}))$$

$$H_d(\tfrac{\pi}{2}) = 1 + e^{-j\frac{\pi}{2}} = 1 - j \Rightarrow \boxed{|H_d(\tfrac{\pi}{2})| = \sqrt{2}}$$

$$\boxed{\angle H_d(\tfrac{\pi}{2}) = \tan^{-1}(\tfrac{-1}{1}) = -45^\circ}$$

$$\Rightarrow y[n] = |H_d(\tfrac{\pi}{2})| \cos(\tfrac{\pi}{2}n + \angle H_d(\tfrac{\pi}{2}))$$

$$\Rightarrow \boxed{y[n] = \sqrt{2} \cos(\tfrac{\pi}{2}n - 45^\circ)}$$

(c) $x[n] = (-1)^n = \cos \pi n$

$\Rightarrow H_d(\pi) = 1 + e^{-j\pi} = 0$

$\Rightarrow |H_d(\pi)| = 0 \quad \angle H_d(\pi) = 0$

$$\boxed{y[n] = 0}$$

$\Rightarrow$ The system is a simple LPF.

(2) $y[n] = x[n] + x[n-10]$

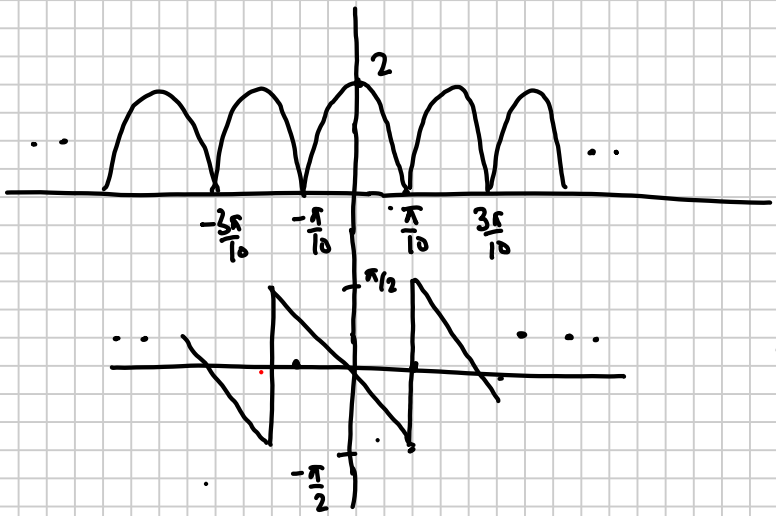
$\Rightarrow Y(z) = X(z) + z^{-10} X(z)$

$\Rightarrow H(z) = 1 + z^{-10}$ Roc includes unit circle

$\Rightarrow H_d(\omega) = 1 + e^{-j10\omega}$

$\Rightarrow H_d(\omega) = e^{-j5\omega}\left[e^{j5\omega} + e^{-j5\omega}\right]$

$$\boxed{H_d(\omega) = 2\cos 5\omega \cdot e^{-j5\omega}}$$



magnitude

&

phase Response.

$x[n] = \cos \frac{\pi}{10}(2n+1)$ gets filtered out

$\rightarrow$ Comb Filter