Recommendations and Cross-selling: Pricing Strategies when Personalizing Firms Cross-sell

Recommender systems enable firms to target customers with products and services that better match their needs, as well as cross-sell products and services. Considering these factors in markets with monopoly and duopoly, we investigate (i) How do pricing strategies differ when firms cross-sell versus when they do not cross-sell, and (ii) How do these pricing strategies change when a firm improves its recommender system? We find that cross-selling can enable a monopolist to subsidize its price for the focal products, while maximizing its profit. In a duopoly, the price set by the firm with the inferior system (low type firm) is always lower when the firms cross-sell than when the firms do not cross-sell; however, that does not necessarily hold for the high type firm. When the high type firm improves its recommender system, the low type firm may decrease its price when firms cross-sell, which does not happen when firms do not cross-sell.

Keywords: recommender system, personalization, duopoly, pricing, cross-selling, game theory

1. Introduction

The value of commerce through online shopping continues to grow. To support online shoppers, retailers are increasingly providing recommendation services to customers (Lee and Hosanagar 2016). For example, *Amazon.com* and Walmart provide recommendations for products such as movies. Recommender systems help customers select the products they desire among myriad choices offered by the retailers by helping customers find products they desire (Resnick and Varian 1997). Researchers have analyzed the influence of recommender systems on consumer search and purchase behavior while shopping. Häubl and Trifts (2000) show that a recommender system reduces the search effort of a customer in gathering information about products. Tam and Ho (2005) find that personalized offers are more likely to get accepted by a customer.

Apart from helping customers find products they desire, recommender systems also enable firms to cross-sell products (as additional recommendations) and generate more revenue. Cross-selling is an activity of selling additional products and services, often complementary, to a customer given that she has purchased a focal product (Kamakura et al. 2004, Bernazzani 2018, Brown and Mehring 2018, Shopify 2019). Once the customer selects a product to purchase (e.g., places an item in the cart and is ready to checkout) the firm offers more products to the customer based on what the firm has learned about her preferences through the items in the cart and her past purchases. For example, if a customer purchases "The Lego Batman Movie" movie from *Amazon.com*, then the customer may be recommended a "Toddler backpack" sold by *Orezi*, a third party seller. Cross-selling benefits the firm by providing extra revenue, as well as the customer by providing her extra surplus from the utility from purchasing the cross-sold product (Akçura et al. 2009). Recently, an online retailer implemented a system for cross-selling based on advice from McKinsey, which led to an increase in revenue by 20% (McKinsey 2017). Similar observations are also made by Senior et al. (2016).

However, it is not clear how cross-selling affects the pricing of the focal products when a firm provides product recommendations. A firm may have an opportunity to increase its price to extract some of the extra surpluses of the customers. Alternatively, since the firm also gets extra revenue from cross-selling, it may consider subsidizing its price of the products to increase demand. In equilibrium, the products should be priced so that these two considerations are balanced. In a competition, these considerations must also account for the reactions of the competitor.

Despite the apparent link between the search process, cross-selling, and recommender systems, little attention has been devoted to study these relationships and their cascading impacts on the demands of focal and cross-sold products, firms' pricing strategies, and eventually their profits. In a monopoly, a firm should change the price of its products based on the changes in search behaviors, and the cross-selling surpluses of its customers, when it improves its system. In a duopoly, a change in its own recommender system may induce a change in the demands of the products of the competitor as well. This is because a changed recommender system effectiveness of the firm may attract some of the competitor's customers and the competitor may react by lowering its own price.

Given that an increasing number of companies are realizing the benefits of implementing recommender systems, analyzing these interrelationships of customer search, purchase behavior, and cross-selling with the firm's pricing strategies have become important. This is the central theme of our research. We consider markets with monopoly and duopoly where customers decide whether to search and purchase products for a given price, and firms decide the price to maximize profit by selling its products and cross-selling. The specific questions that we analyze are the following: (i) How do pricing strategies differ when firms cross-sell as compared to when they do not cross-sell, and (ii) when firms cross-sell, how do prices change when a firm improves its recommender system? We also study the implications of these pricing strategies on the profits of the firms and on customer surpluses.

1.1. Problem Setting

We model a game between customers, heterogeneous in search costs and with full information, and electronic market places (firms) where customers shop to purchase products offered by the firms. The customers have apriori knowledge from past experiences about the recommender system effectivenesses and cross-selling surpluses.[1] The firms set the prices first for given (exogenous) recommender system effectivenesses, and customers react to the prices when making a purchase decision. For example, a customer who wants to purchase an action movie (focal product) could visit *Amazon.com*

[1] Pathak et al. (2010) and Formisimo.com (2019) note that customers recognize the benefits they obtain from recommender systems and cross-selling, respectively.

and examine movies offered in the category of interest. The firm (*Amazon.com*) could recommend various movies that are potentially close to the ideal[2] item sought by the customer. In addition, *Amazon.com* may also cross-sell products sold by third parties (such as toys, bags, associated books, etc). Customers expend effort in searching for products and the recommender system alleviates the effort spend by the customer. Customers purchase products when then expect a non-negative surplus. Thus the effectiveness of the firm's recommender system influences the number of customers who purchase products from the firm and hence, the firm's profit. Further, the customers may purchase cross-sold products (i.e., offers of other products sold by the firm or its partners) that may be of interest to her.[3] We allow the cross-sold products to be any product sold online so long as they provide a positive revenue to the firm (either directly or from a third-party seller), and a potentially positive surplus to the customer. If the customer purchases a cross-sold product, she enjoys a surplus and the firm obtains additional revenue.

The game is solved backwards, starting with the customers' search and purchase decisions, followed by the firms' decisions of setting the prices. Every customer has a reservation utility for the products. Customers who search and examine products incur three costs. The first cost she incurs is a *search cost* from the effort exerted in the search process; this cost increases in the effort. Second, the customer incurs a *mismatch cost* which is the opportunity cost borne by the customer from potentially purchasing a non-ideal product. This cost decreases in the effort of the customer and the effectiveness of the recommender system. Finally, the customer must pay the price of the product. The customer obtains a surplus from purchasing a cross-sold product (hereafter referred to as

[2]An ideal product is the one which a customer would like to select after examining (hypothetically) every single product offered by the firm.

[3]Although products that are offered for cross-selling are likely to complement the focal product, such product characteristics are not relevant for our model setup.

the customer's *cross-selling surplus*), which increases with a better recommender system. The expected surplus of a customer is the aggregate of the reservation price of the focal product (i.e., the willingness to pay amount) and cross-selling surplus minus the three costs. For the customer to make purchases, this expected surplus should be non-negative.

We first analyze a monopolist who sells its products to customers for a price that maximizes its profit. Next, we consider a duopoly where firms are asymmetric in their recommender system effectivenesses. The firm with the superior recommender system effectiveness is referred to as the high type firm, and the other as the low type firm. Each customer either purchases a product from the firm that provides her a higher expected positive surplus or does not purchase from either firm. Based on demand, the firms decide prices through a simultaneous move price game that maximize their profits in equilibrium. In both the monopoly and duopoly scenarios, we compare the pricing strategies when the firms cross-sell with when the firms do not cross-sell. Further, we study changes in the pricing strategies when the firms improve their recommender systems while cross-selling. We consider focal products within a category, and assume that a firm prices all the products the same within the category.

1.2. Summary of Main Results

We analyze how the search effort, optimal price, market size, and customer surpluses differ between situations when firms cross-sell and when they do not, and how they change when the recommender system improves, first for a monopoly and then for a duopoly. Below we highlight the important findings of this research.

- We find that when a monopolist cross-sells its own products, the pricing strategies do not change when it does not cross-sell. This is intuitive because the cross-sold products are similar to the focal product from the pricing strategy point of view – in other words, cross-selling own products is like selling more of the focal products. However, the pricing strategies change for the focal products when a monopolist cross-sells products of third parties (i.e., when the revenue obtained by the firm from cross-selling is exogenous). Compared to when the monopolist does not cross-sell, it charges a lower price for the focal product when it cross-sells if the expected cross-selling surplus of the customers is lesser than a threshold. This is because the firm obtains extra revenue through cross-selling which enables it to subsidize the price of the focal product and thereby increase the demand of both the focal and the cross-sold products. This can potentially increase the profit as compared to when the subsidy is not provided. However, when the expected cross-selling surplus of a customer is greater than the threshold, the firm finds it optimal to extract a portion of the cross-selling surplus of the customer by increasing the price. We also find that when some customers purchase the first product recommended without expending additional search effort in examining other products, the firm may decrease the price under certain conditions when it improves its recommender system.

- In a duopoly when both firms cross-sell their own products, the pricing strategies remain the same as when they do not cross-sell. When they cross-sell products of third parties, we find that the low type firm always sells products at a lower price compared to when the firms do not cross-sell; however, if the customers' expected cross-selling surplus is more than a threshold value, the high type firm charges a higher price when the firms cross-sell than when they do not. This is interesting because under this condition, the low type firm bears the pressure of competition and decreases its price when it cross-sells products, but the competitive pressure is not intense on the high type firm, and the firm profits more by charging a higher price for its products while cross-selling.

- When the high type firm improves its recommender system and the firms cross-sell third party products, the firm always increases its price. However, the low type firm may decrease its price despite the increase in the differentiation between the two firms. This unexpected phenomenon occurs when the extra cross-selling surplus (from the improved system of the high type firm) leads to customers of the low type firm switching in large numbers to the high type firm. The low type firm reacts by decreasing its price in order to reduce the scale of migration; however, this phenomenon never happens when firms do not cross-sell. This result is contrary to expectations from prior literature in product quality (Moorthy, 1988), and demonstrates that the ability to cross-sell on online platforms require unique pricing strategies that cannot be inferred from the existing literature.

The rest of the paper is organized as follows. Related research is discussed in Section 2. Section 3 describes the basic setup of the model. In Section 4, we develop and analyze a model for a monopolist. In Section 5, we analyze a market with a duopoly. Section 6 extends the model to the case when some customers do not search and rather purchase the first product recommended to them. Finally, Section 7 provides managerial insights and concludes the paper.

2. Literature Review

Our work is related to the streams of literature on personalization, product customization, vertical differentiation, and search costs. Since our main contribution is in the area of personalization and recommender systems, we discuss that literature first. Researchers have extensively analyzed the value recommender systems generate for customers using empirical techniques. Kumar and Benbasat (2006) show that providing personalization services improve the perceived usefulness of the website. Pathak et al. (2010) demonstrate increased sales when firms provide recommendations. In a similar vein, Zhang et al. (2011) show that higher quality product recommendations are associated with greater values derived by customers from a website that provides personalizing services, and is also positively associated with repurchase intentions. Thirumalai and Sinha (2013) delve into customer loyalty and personalizing services provisions. They find that a firm that self-selects to provide personalization services typically observes a greater customer loyalty. On the other hand, using counter-factual analysis they show that if the firm that self-selects not to provide recommendations had chosen to provide recommendations, it would not have observed customer loyalty at the same level as the ones who self-selected to provide recommendations. Jabr and Zheng (2014) show that recommendation services intensify competition among products when customer reviews are also provided. None of these papers investigate a competitive setting between recommending firms, nor do they consider cross-selling. Wattal et al. (2010) consider the pricing decisions of two firms when they provide recommendations. Their work primarily focuses on the interaction of recommendation services and general qualities of goods provided by firms when firms differ in both these dimensions. They find that when firms differ in personalization effectivenesses and the costs of providing higher quality products are high, firms that provide better recommendations also offer higher quality products. However, the effects of cross-selling on pricing strategies cannot be inferred from that study.

Several recent papers have focused on operational aspects of recommender systems. Liu et al. (2010) studies the trade-off faced by a content-delivery website: either to deliver a superior personalized content with some delay (possibly), or to deliver an inferior version quickly. They propose a batch scheduling scheme that balances this trade-off to maximize revenue. Johar et al. (2014) considers a problem where a profit maximizing firm decides what proportion of a set of products shown to a customer (an offerset) should be targeted towards learning the preferences of a customer verses generating immediate sales. They find conditions under which the offerset should be geared towards profiling rather than selling, and vice-versa. Fleder and Hosanagar (2009) examine the effect of recommender systems on consumer choices: whether a recommender system helps consumers find new products, or whether it only reinforces the popular products. Using a simulation approach, they establish that both can occur under different conditions. Our paper is distinct from these papers since these papers do not consider the economic effect of customer search and cross-selling opportunities on product prices.

Several marketing researchers, such as Chen and Iyer (2002) and Chen and Zhang (2009) derive insights based on a firm's ability to target customers based on the customers' preferences. However, the modeling setup they consider are different from ours, and the papers do not provide any guidance on how to price products when firms improve their recommender system and when they cross-sell products.

In our model with duopoly, firms may be visualized as vertically differentiated in terms of the qualities of recommendation services they provide. They are also differentiated in terms of the preferences of the customers towards them (i.e., customers are segmented based on the firm they prefer to purchase products from depending on their costs). The literature in this area is vast, and we discuss only the prominent literature whose models bear some similarities with our model. Shaked and Sutton (1982) consider an oligopolistic market with customers heterogeneous in their incomes. Ferriera and Thisse (1996) and Wauthy (1996) consider vertically differentiated firms and heterogeneous customers. Bhargava and Choudhary (2001) model a monopolistic firm offering products with different qualities. Moorthy (1988) considers the competition between firms when customers have heterogeneous utilities for product qualities.

Our research is quite distinct from this stream of research because our modeling setup has sharp differences from the ones discussed above, stemming from the online context that we consider. First, none of these papers consider a decision variable analogous to the effort of the customer, and this has important implications. The heterogeneity in a customer's cost depends on the amount of effort she employs in searching for the product to purchase. Hence, the customer is able to control her cost. Thus, while at some level of effort, the customer may prefer to purchase a product from one firm, at another level of effort she may prefer another firm. Consequently, the models studied in the above literature are not suitable to solve our problem because search effort is not considered in these models. Second, we also model cross-selling and find results that either cannot be inferred or are counter to the expectations from the past literature. We find that cross-selling revenue can subsidize the prices of products in both a monopoly and a duopoly which leads to more nuanced firm behaviors. For instance, when the high type firm improves its system, the presence of cross-selling surplus can make the firm even more attractive to customers, leading to its competitor (the low type firm) reducing its price in order to retain customers. Also, when the low type firm improves its system, the market sizes of the firms reduce under certain conditions, although not simultaneously. Further, when the high type firm improves its system, the market size of the low type firm can increase and that of the high type firm can decrease under certain conditions. These phenomena are not observed in the quality literature and are novel in our context.

3. Model Preliminaries

We consider a setting where a firm sells a vast array of products under a category, e.g., movies of different genres within the movies category, and sets a price for the category.[4] A customer typically explores a number of products recommended to her in order to eventually purchase one (Moe and Fader 2004, Poesler 2018).[5] The exploration may involve reading product descriptions and reviews, checking the ratings and popularity, sampling the products (e.g., listening to short samples of songs or watching movie trailers), etc. Overall, the surplus expected by a customer from visiting the firm, accounting for the costs she expects to incur in order to find an acceptable product,

[4]If the firm sells products from multiple categories, the model can be applied to each category.

[5]If the customer knows the exact details of the product she intends to purchase, then exploration or recommendations are not necessary – she can just purchase it, if available.

should be positive for the customer to initiate a session. Following are the factors that determine the expected surplus of the customer.

Reservation Price (R): This is a customer's willingness to pay for a product within a category, and is assumed to be the same for all the customers and for all the products within the category.

Price of the product (p): Since the products belong to the same category, in line with the existing literature, we assume that all products have the same price (Wattal et.al. 2009). This assumption is largely consistent with observations in companies like *Amazon.com* which offers a considerable collection of movies for around $4.99, with relatively small variations in prices.

Search cost: This is the cost incurred by the customer for the effort she exerts when exploring the products while shopping. We denote the effort of the customer by y. This effort includes examinations of choices such as reading the reviews of recommended products, partially listening to songs, watching movie trailers, etc. As a customer must at least visit the site even before navigating through the website during the search, the effort $y > 0$. The search cost to a customer is positive when $y > 0$. As mentioned earlier, the customers are assumed to be heterogeneous in their search costs, i.e., the customers incur different search costs for the same amount of effort. A normalized parameter $\theta \in [\alpha, 1]$, where $0 < \alpha < 1$, is used to characterize a customer and is referred to as the search cost parameter of the customer (i.e., θ denotes the search cost to the customer per unit effort).[6]

We borrow the functional form of the search cost from Janssen and Morga-González (2004) who model it as the product of the number of inquiries made by the customer and

[6] When $\theta = 0$, the customer has zero search cost for any amount of effort. Then, she would search all the products and find her ideal product to incur zero mismatch cost. For such a customer, the presence of a recommender system does not impact her costs at all. Hence, this is an uninteresting case.

the search cost per unit effort. The inquiries made by the customer is analogous to examining the products in the consideration set, which is directly proportional to the effort she exerts (i.e., y). As θ represents the search cost per unit effort, the total search cost is $A\theta y$, where A is a positive scaling factor. We note that, prior to beginning a session, a customer may not know the exact amount of effort she will spend in searching for a product. Further, the actual search effort invested in identifying a product in one session would typically be different from that invested to identify another product in a different session, and the search process may even span multiple sessions. When deciding whether to engage in a search, a customer would base this decision on her estimate of the expected effort needed to identify an acceptable product. Consequently, the expression $A\theta y$ represents the total expected search cost for a successful purchase.

The probability density function for θ, $\psi(\theta)$, is assumed to be decreasing and convex like the Pareto distribution (or power law), i.e, a customer has a higher probability of having an associated low search cost parameter than a high search cost parameter (Tarascio 1973). The Pareto distribution represents well the distribution of income (or the opportunity cost for searching), and the cost of search is expected to be proportional to the opportunity cost. The distribution of θ is known to the firm but θ's of individual customers are not.

Mismatch cost: Usually, the number of choices available to the customer is enormous and the customer does not have the cognitive resources to examine all of them (Beach 1993). Hence, the customer may end up purchasing a product that is not her ideal product, thereby incurring a mismatch cost. The mismatch cost depends upon the effort, y, spent by the customer in examining additional products and the recommender system effectiveness; i.e., the recommender system influences the effort exerted by the customer. Häubl and Trifts (2000) show through a carefully designed experiment that users of a recommender system end up purchasing better quality products than those who do not use a recommender system for the same amount of effort, where the quality of the chosen product is measured in terms of the fitness with their needs. This indicates that recommender systems alleviate the effort exerted by the customer to find a product of equal fitness. In other words, the effort of the customer interacts with the recommender system; we refer to the interaction as *effective effort*. Increased effective effort should help the customer find a product with a lower mismatch cost; i.e., mismatch cost will be a decreasing function of the effective effort.

The rate at which the mismatch cost decreases with an increase in effective effort is expected to diminish. As the customer examines more recommended products, i.e., exerts more effective effort, it becomes increasingly difficult for her to find a product that is even closer to her ideal product and reduce her mismatch cost further. This is because the recommender system starts by recommending those products whose likelihoods of being selected by the customer are high based on the customer's profile (Deshpande and Karypis 2004). If the customer wishes to examine more products, the products subsequently recommended usually have smaller probabilities of being selected as compared to those recommended earlier. Therefore, although examining more products usually enables her to find one that is closer to her ideal product and reduces her mismatch cost, the cost reduces at a slower rate as the expected relevance of the products subsequently recommended to the customer reduces.

Consequently, we model the mismatch cost to be decreasing and convex in effective effort using the functional form $\frac{B}{\text{effective effort}}$, where B is a positive scaling factor. This expression captures the desirable properties discussed above. Analytically, the effective effort should have the property that the recommender system effectiveness compensates for the marginal effort expended by the customer since a better recommender system should help the customer find a product closer to her ideal product for the same amount of effort of the customer. We represent the effective effort as yr^2 (the quadratic form in r

helps in analytical tractability). We assume that $r = 1$ for a firm with no recommender system (the maximum mismatch cost possible), and to be consistent with our setup that requires the presence of a functioning recommender system, we assume $r > 1$. For any valid value of r, the mismatch cost decreases with effort, and increasing the value of r leads to a reduction in mismatch cost (i.e., a better recommender system decreases the mismatch cost). The sum of search cost and mismatch cost is referred to as the *product acquisition cost*.

Cross-selling surplus: The search process of the customer and the final purchase decision reveals useful information to the firm about the preferences of the customer (Akçura and Srinivasan 2005). A firm can leverage this information and sell more products by cross-selling, e.g., Sainsbury makes cross-selling offers for selling music online (Baker 2012). Cross-selling may also occur through advertisements of products sold by partner firms (with referral fees accruing to the focal firm).[7] For example, at *Amazon.com*, many products that appear for cross-selling are sold by third party merchants who pay a percentage of sales price to *Amazon.com* (Amazon 2019). The cross-sold products are usually the ones that the customer had not searched for explicitly, but may be interested in given the products searched and items purchased. Such products are predictions made by the recommender system, and often complement the main products purchased by the customer, potentially providing a positive surplus to the customers (Akçura et al. 2009).

We assume that the customer purchases the cross-sold product if she expects to obtain a positive surplus from the product. Customers do not necessarily purchase a cross-sold product every time it is displayed to the customer (for example, the product may not be a good fit for the customer despite the recommendation; or even if the product is a good fit, she may not have the budget to purchase the product). For tractability, we denote the

[7] We also consider the possibility that the firm cross-sells its own products; however, the primary focus is on the case when the firm cross-sells third party products because the insights are new and particularly interesting for that case.

expected cross-selling surplus of the customer by βr where $\beta(>0)$ is the cross-selling parameter—a long-run average marginal benefit from cross-selling derived by a customer per unit recommender system effectiveness. Different possibilities, such as the customer incurring a mismatch cost from the cross-sold product (similar to the mismatch cost for the focal product) or the customer foregoing the cross-sold product, are incorporated in βr.

On the firm's side, suppose N denotes the total number of potential customers in the market. Let σ denote the expected cross-selling revenue per customer, which the firm obtains in expectation from selling the cross-sold products while selling the focal product. We assume that the firm knows how the costs of customers (mismatch cost and search cost) vary with θ, y and r. Table 1 presents all the parameters.

⟨Insert Table 1 here.⟩

In the subsequent sections, we show how these factors impact the decisions of the customers and firms.

4. Monopoly

We solve the full model backwards by first determining the purchase decision of the customers followed by the profit maximizing price of the product. We first consider customers who are willing to expend effort in searching and examining one or more products before purchasing.[8] The expected surplus (which is a function of the effective effort needed to identify an acceptable product) of a customer is

$$R - A\theta y - \frac{B}{yr^2} - p + \beta r.$$

[8]In practice, some, perhaps impulsive, customers purchase the first product displayed. In Section 6, we enhance our model to include such customers in our analysis.

The customer expects to exert a surplus maximizing effort to find a product to purchase; this effort is $y^* = \frac{1}{r}\sqrt{\frac{B}{A\theta}}$. The optimal expected surplus expression for a customer then becomes

$$S^* = R - \frac{2\sqrt{AB}\sqrt{\theta}}{r} - p + \beta r. \quad (1)$$

It is clear from Equation (1) that S^* monotonically decreases with the customer's search cost parameter θ. Thus, customers with low θ values always obtain higher optimal surpluses than the customers with relatively higher θ values. As expected, the highest amount of effort is spent by the customer with $\theta = \alpha$, i.e., the customer with the lowest search cost parameter. We assume that $R - \frac{2\sqrt{AB}\sqrt{\theta}}{r} + \beta r > p$ to ensure that the firm has a market to sell its products (i.e., $S^* \geq 0$ for some customers at least). When a customer expects the maximum surplus to be positive ($S^* \geq 0$), she engages in a search and potentially purchases. Otherwise, she does not search or purchase anything from the firm (the *individual rationality* constraint). A customer with the search cost parameter $\theta = \bar{\theta}$ is called a *marginal customer* if customers with $\theta > \bar{\theta}$ do not search (or buy). The value of $\bar{\theta}$ characterizes the market. If $\bar{\theta} < 1$ for a given price, then the market is divided into two segments – purchasers ($\theta \in [\alpha, \bar{\theta}]$) and non-purchasers ($\theta \in (\bar{\theta}, 1]$). In a similar manner, the maximum surplus of a customer and the value of $\bar{\theta}$ (and thus, the range of θ's of the purchasers) can be obtained for the case when the firm does not cross-sell; the discussion is provided in Section B of the Appendix.

Some properties of the market size are apparent from the expression of S^*. When the firm improves its recommender system without increasing the price, the optimal efforts exerted by the existing customers, and consequently their search costs and mismatch costs, decrease. Also, some new customers start purchasing products. Hence, the market size increases. On the other hand, if the firm increases price without changing the recommender system effectiveness, the market size decreases. Thus, the firm should re-optimize its price when it changes its recommender system effectiveness.

4.1. Profit of the Firm

The firm maximizes its profit by selecting the optimal price. For customers, we assume the following probability density function $\psi(\theta) = \frac{k}{\sqrt{\theta}}$ as this form demonstrates the desirable properties discussed above.[9] Since, $\int_{\alpha}^{1} \frac{k}{\sqrt{\theta}} d\theta = 1$, $k = \frac{1}{2\left(1-\sqrt{\alpha}\right)}$. Thus, $\psi(\theta) = \frac{1}{2\left(1-\sqrt{\alpha}\right)} \frac{1}{\sqrt{\theta}}$. The revenue of the firm from sales of products is $pD(p,r)$, where $D(p,r)$ is the demand of the product. The demand originates from the customers who intend to search and purchase, i.e., $D(p,r) = \frac{N}{2\left(1-\sqrt{\alpha}\right)} \left(\int_{\alpha}^{\bar{\theta}} \frac{1}{\sqrt{\theta}} d\theta\right)$. Therefore, the revenue of the firm from such sales is $\frac{Np}{2\left(1-\sqrt{\alpha}\right)} \left(\int_{\alpha}^{\bar{\theta}} \frac{1}{\sqrt{\theta}} d\theta\right)$. The revenue from cross-selling is $\sigma D(p,r)$.[10]

The costs incurred by the firm include the cost of developing the infrastructure for storing, analyzing, and maintaining the customer data (Leavitt 2006). The cost of developing infrastructure is incurred only once. Some other one time costs may be incurred by the firm while improving the system. We assume that the firm has the capital for building the infrastructure, bears the one time costs, and the total cost is sunk. Finally, the marginal cost of personalization for information goods is often negligible (Wattal et. al. 2009). Thus, we ignore the cost of providing recommendations and focus on the revenue of the firm.

Therefore, the total profit (revenue) of the firm is

$$\Pi = \frac{N(p+\sigma)}{2\left(1-\sqrt{\alpha}\right)} \int_{\alpha}^{\bar{\theta}} \frac{1}{\sqrt{\theta}} d\theta = \frac{N(p+\sigma)}{\left(1-\sqrt{\alpha}\right)} \left(\sqrt{\bar{\theta}} - \sqrt{\alpha}\right). \tag{2}$$

[9] We have also analyzed our results for $\theta \sim U[0,1]$. Our results regarding the pricing strategies remain qualitatively similar for both monopoly and duopoly.

[10] When the firm cross-sells its own products, the results remain qualitatively similar to when the firm does not cross-sell. The solution is provided in the Appendix in Section F.1.1 and F.2.1.

From Equation (2), we observe that the profit of the firm increases if price increases provided the size of the market does not change. However, $\bar{\theta}$ decreases when p increases. Likewise, an increase in the recommender system effectiveness (r) increases the firm's market size but may also justify an increase in the price. Hence, the firm should balance the trade-off between price and market size by choosing the optimal price (p^*) to maximize its profit. The optimal price and profit are (the derivation is provided in Section B of the Appendix):

$$p^* = \frac{1}{2}\left(R - \frac{2\sqrt{AB}\sqrt{\alpha}}{r} + \beta r - \sigma\right) \text{ and } \Pi^* = \frac{N(r(R + r\beta + \sigma) - 2\sqrt{AB\alpha})^2}{8r\sqrt{AB}(1 - \sqrt{\alpha})}.$$

For an equilibrium to exist, the parameters A and B should satisfy the following bounds:

$$\frac{r(R + \beta r + \sigma)}{2(2 - \sqrt{\alpha})} < \sqrt{AB} < \frac{r(R + \beta r + \sigma)}{2\sqrt{\alpha}}. \tag{3}$$

This expression is derived using $\alpha < \bar{\theta} < 1$, where $\bar{\theta} = \left(\frac{r(R+\beta r+\sigma)+2\sqrt{AB\alpha}}{4\sqrt{AB}}\right)^2$; see Section B of the Appendix for the derivation. Proposition 1 summarizes the effects of cross-selling on price, market size, and customer surplus (Section B of the Appendix also provide the expressions for the case when the form does not cross-sell).

Proposition 1. *Compared to the case when firms do not cross-sell, the firm charges a higher price if $\beta r > \sigma$ when it cross-sells. The market size and the surpluses of the individual customers are always higher when the firm cross-sells.*

All proofs are provided in Section D of the Appendix. The difference between the prices of the products when the firm cross-sells and when it does not is $\frac{\beta r - \sigma}{2}$. Therefore, when the customer obtains a higher surplus from cross-selling than what the firm obtains as revenue per unit demand from cross-selling (i.e., $\beta r > \sigma$), the firm increases its price to extract a portion of the surplus. Otherwise, the firm decreases its price. An additional surplus of $\frac{\beta r + \sigma}{2}$ accrues to all the customers (individually), leading to higher surpluses for all when the firm cross-sells. As a result, a marginal customer in the case of the firm not cross-selling obtains additional surplus when the firm cross-sells, leading to a higher market size when the firm cross-sells. When the firm lowers its price (i.e., $\beta r < \sigma$), expectedly, the market size and the surpluses of all customers again increase.

Upon analyzing the equilibrium in which the firm cross-sells, we find that as the customer's expected cross-selling surplus parameter (β) increases, the firm increases its price in order to extract a portion of the increased surplus of the customer. The remainder of the surplus is left with the customer, leading to an increase in market size. When σ increases,[11] the firm finds it optimal to decrease the price to pass down some of the additional revenue to the customers. The market size increases, as expected.

When the firm improves its recommender system, the price increases, to again extract a portion of the extra surplus that is generated by the reduction in the product acquisition cost. The rest of the additionally generated surplus goes to the customer, which leads to an increased overall customer surplus and market size. This result may have implications for firms like *Amazon.com* in markets such as e-books (Amazon holds 83% market share compared to any other company for selling e-books without ISBN numbers (publishdrive 2020)). By improving its recommendation quality for e-books, it can charge a higher price while also increasing its market size.

5. Duopoly

The market consists of two firms (Firm 1 and Firm 2); these two firms have recommender systems of effectivenesses r_1 and r_2, respectively, and prices p_1 and p_2, respectively. Without loss of generality, we assume $1 < r_1 < r_2$ (later, we discuss why firms will remain asymmetric in the equilibrium in terms of the effectivenesses), i.e., Firm 1 is a *low* type firm and Firm 2 is a *high* type firm. We consider a market where customers either

[11] The third parties would be naturally motivated to improve their service offerings and qualities of products since they will be able to sell more of their products.

purchase from one of the firms, or do not purchase anything at all (a general setup). Customers consistently purchasing from one firm is not uncommon; e.g., Winer (2007) reports that more than 80% of customers do not shop around for retail products, such as books, electronics etc., and visit the same online store repeatedly to purchase products. Thus a customer transacts with the firm from which she expects to obtain higher surplus. The sequence of the game is the following: Firms simultaneously decide the prices of their products. Depending upon the given prices, costs, and cross-selling surpluses, individual customers determine their surpluses and decide to purchase from the firm that provides them higher non-negative surpluses. The highest surplus obtained at the optimal effort with firm $i \in \{1,2\}$ is

$$S_i^* = R - \frac{2\sqrt{AB}\sqrt{\theta}}{r_i} - p_i + \beta r_i. \tag{4}$$

We next discuss some of the general properties of the market with duopoly. We define θ_s as the search cost parameter of an *indifferent* customer who obtains the same surplus from both firms. The search cost parameter θ_s has a unique value based on the Spence-Mirrlees single crossing condition. Using Equation (4), we find $\frac{dS_i^*}{d\theta} = -2\frac{\sqrt{AB}}{\sqrt{\theta}r_i} < 0$; therefore, the rate of decrease of S_2^* with respect to θ is always less than the rate of decrease of S_1^* with respect to θ $\forall$ $\{r_i, \theta\}$. Thus, S_1^* and S_2^* intersect at most once, i.e., there is only one possible $\theta = \theta_s$ (Rasmusen 2007). Likewise, the value of θ at which $S_i^* = 0$ is also unique (as discussed below, $S_2^* = 0$ corresponds to the marginal customer who purchases products from the high type firm; we denote her search cost parameter by $\bar{\theta}_2$). Based on these observations, Lemma 1 and Figure 1 summarize the structure of the market (the structure applies to both types of scenarios - firms cross-selling as well as not cross-selling).

LEMMA 1. *The low type firm charges a strictly lower price than the high type firm. Customers with search cost parameters $\theta \in [\alpha, \theta_s]$ purchase products from the low type* *firm and customers with search cost parameters* $\theta \in [\theta_s, \bar{\theta}_2]$ *purchase products from the high type firm.*

⟨Insert Figure 1 here.⟩

From Equation (4), it follows that the low type firm should charge a lower price than the high type firm for its products, as otherwise no customer can expect to obtain a higher surplus from the low type firm (since $r_1 < r_2$). Lemma 1 implies that there are three segments of customers: customers with $\theta \in [\alpha, \theta_s]$ who purchase products from the low type firm, customers with $\theta \in \left[\theta_s, \bar{\theta}_2\right]$ who purchase products from the high type firm, and customers with $\theta \in (\bar{\theta}_2, 1]$ who do not purchase any product. This is intuitive since a customer with a low search cost parameter is willing to expend a higher amount of effort to obtain a suitable product and purchase it at a lower price. On the other hand, a customer with a high search cost parameter relies more on the effectiveness of the recommender system, expending less effort to find the desired product and paying a higher price for it. Further, the surplus decreases at a decreasing rate with θ (Equation 4). The term $\frac{1}{r_i}$ affects the rate at which the surplus decreases; therefore, since $r_1 < r_2$, S_1^* decreases at a faster rate than S_2^* with increase in θ. Finally, an increase in the price reduces the surplus linearly.

For an indifferent customer, $S_1^* = S_2^*$, and for a marginal customer, $S_2^* = 0$. Hence,

$$\theta_s = \left(\frac{(p_2 - p_1 + \beta\,(r_1 - r_2))\,r_1 r_2}{2\sqrt{AB}\,(r_2 - r_1)}\right)^2 \text{ and } \bar{\theta}_2 = \left(\frac{(R - p_2 + \beta r_2) r_2}{2\sqrt{AB}}\right)^2. \tag{5}$$

The above expressions are for the case when firms cross-sell; Section C of the Appendix provides the relevant expressions for the case when firms do not cross-sell. If the prices and recommender system effectivenesses are such that the customer with search cost parameter θ_s lies in $[\alpha, 1)$, both firms will co-exist, i.e., the market has a duopoly. When $\alpha < \theta_s < 1$ does not hold, two scenarios are possible: (i) $\theta_s \leq \alpha$, and (ii) $\theta_s \geq 1$. In the first scenario, customers purchase only from the high type firm, whereas, in the second scenario, customers purchase products only from the low type firm. Both scenarios correspond to a monopoly. We have already shown in Lemma 1 that $\theta_s < \bar{\theta}_2$. If $\theta_s < 1 < \bar{\theta}_2$, then the low type firm sells products to customers with low search costs (with $\theta < \theta_s$) and the rest of the customers purchase from the high type firm. The most general case is when $\bar{\theta}_2 < 1$, i.e., when some customers do not purchase products from either firm (also a typical scenario in the e-commerce market). In summary, the condition $\alpha < \theta_s < \bar{\theta}_2 < 1$ characterizes a market where both firms sell products and some customers do not purchase. This condition is used to derive the parameter range in which a duopoly exists (details are in Section C of the Appendix).

$$\frac{(R - p_2 + \beta r_2)\, r_2}{2} < \sqrt{AB} < \frac{(p_2 - p_1 - \beta(r_2 - r_1))r_1 r_2}{2\sqrt{\alpha}(r_2 - r_1)}. \tag{6}$$

Firms evaluate the sizes of their markets from the distribution of the search cost parameters of customers. They choose prices for their products that maximize their profits in the equilibrium.

5.1. Profits of Firms

The firms simultaneously decide the prices of their products in equilibrium using the following profit expressions when the cross-selling revenue is generated by selling the products of the third parties[12]:

$$\begin{aligned}
&\Pi_1 = D_1(p_1, p_2, r_1, r_2)(p_1 + \sigma_1) \text{ and } \Pi_2 = D_2(p_1, p_2, r_1, r_2)(p_2 + \sigma_2), \text{where} \\
&D_1(p_1, p_2, r_1, r_2) = \int_{\alpha}^{\theta_s} \frac{N}{2(1-\sqrt{\alpha})\sqrt{\theta}} d\theta = N\frac{(\sqrt{\theta_s} - \sqrt{\alpha})}{1-\sqrt{\alpha}} \text{ and} \\
&D_2(p_1, p_2, r_1, r_2)p_2 = \int_{\theta_s}^{\bar{\theta}_2} \frac{N}{2(1-\sqrt{\alpha})\sqrt{\theta}} d\theta = N\frac{\left(\sqrt{\bar{\theta}_2} - \sqrt{\theta_s}\right)}{1-\sqrt{\alpha}}.
\end{aligned}$$

[12] When the firms cross-sell their own products, the results are qualitatively the same as when the firms do not cross-sell. The solution is provided in the Appendix in Sections F.1.2 and F.2.2.

The firms play a simultaneous move price game to decide their prices.[13] The derivation of the equilibria (for both cross-selling and no cross-selling) are provided in Section C of the Appendix. We determine the equilibrium prices (p_1^* and p_2^*) and market sizes; M_1^* and M_2^* are equilibrium market sizes of the low type firm and the high type firm, respectively, and $M^* = M_1^* + M_2^*$ is the total market size. Our setup is similar to a Bertrand Game with the recommender system effectiveness being the source of differentiation, and the firms make positive profits as long as they remain differentiated based on their recommender system effectivenesses, i.e., $r_1 \neq r_2$. Thus, the existence of the differentiated recommender systems is essential for this duopoly to exist, and when the low quality firm has the means to improve its system, it always avoids making its recommender system effectivenesses equal to that of the competitor.

We derive the boundary conditions from the equilibrium results by substituting the equilibrium prices in Equation (6). These conditions also ensure that the prices and market sizes are positive.

$$\frac{Rr_2(2r_2+r_1)+r_2(\beta(r_1^2+2r_2^2)+\sigma_1 r_1+2\sigma_2 r_1)}{2\sqrt{\alpha}((4r_2-r_1)-2\sqrt{\alpha}(r_2-r_1))} < \sqrt{AB} < \min\left\{ \begin{array}{c} \frac{Rr_1(r_2-r_1)-\beta\left(r_1^3-2r_1^2r_2+r_1r_2^2\right)-\max\{(r_1^2-2r_1r_2)\sigma_1,2r_1r_2\sigma_1\}+r_1r_2\sigma_2}{4(r_2-r_1)}, \\ \frac{2Rr_2(r_2-r_1)-\beta\left(r_1^2r_2-r_1r_2^2-2r_2^3\right)-r_1r_2\sigma_1-\max\{(r_1r_2-2r_2^2)\sigma_2,2r_2^2\sigma_2\}}{2(r_2-r_1)} \end{array} \right\} \tag{7}$$

We first discuss the pricing strategies of the two scenarios, when firms cross-sell and when they do not. When firms cross-sell, on one hand, the firms may reduce prices since they obtain extra revenues from cross-selling. On the other hand, they may want to increase their prices to extract some of the additional surpluses that the customers obtain from cross-selling. Unlike a monopoly, now the firms also must account for the reactions of the

[13]We do not consider a game where the expected cross-selling revenues are strategic variables. As mentioned earlier, these revenues may be realized from profit sharing contracts with partner firms, or from showing advertisements and getting paid according to pay-per-impression or pay-per-click. Involvement of partner firms reduces the flexibility in controlling these revenues, making analysis of such a game less appealing for deriving practical insights.

competitors. A decrease in price of one firm may prompt a decrease in price of the competitor, thereby limiting the increase in the demand (and the consequent positive effect on the profit) of the former. We find that the equilibrium prices and market sizes when firms cross-sell differ in unexpected ways from when the firms do not cross-sell.

The following proposition characterizes how the pricing strategies differ when firms cross-sell as compared to when they do not. Let $b_{CS,h} = \frac{r_1\sigma_1 + 2r_2\sigma_2}{(r_2 - r_1)(2r_2 + r_1)}$, $C_{CS,h} = \frac{r_2}{r_1}(r_1 + 2r_2)(r_2 - r_1)\beta + 2(\frac{r_2}{r_1}) - 1$, and $C_{CS,l} = 2 - \frac{1}{r_1}(1 - \frac{r_1}{r_2})^2\beta - \frac{r_1}{r_2}$. These expressions provide the thresholds when the prices and market sizes increase or decrease, and they depend on the relative values of the cross-selling revenues and recommender system effectivenesses.

Proposition 2. *As compared to when firms do not cross-sell,*

(a) When firms cross-sell, the price of the low type firm is always lower; the price of the high type firm is higher iff $\beta > b_{CS,h}$.

(b) The market size of the high type firm (low type firm) is higher iff $\frac{\sigma_1}{\sigma_2} < C_{CS,h}$ $\left(\frac{\sigma_2}{\sigma_1} < C_{CS,l}\right)$ *when firms cross-sell.*

Proposition 2 summarizes how the prices and market sizes differ when firms cross-sell from when they do not. Cross-selling provides extra revenue to customers which decreases competition intensity because cross-selling surplus from the low type firm is smaller than that from the high type firm ($\beta r_1 < \beta r_2$). In such a scenario, an increase in prices of both firms is expected in order to exploit the decreased intensity of competition. Instead, we find that the high type firm alone increases its price, and only when $\beta > b_{CS,h}$. Under this condition, the cross selling surpluses that customers obtain from the high type firm becomes significantly larger than that from the low type firm, thereby enabling the former to increase profits despite some customers switching to the low type firm. Otherwise, both the high type and the low type firms subsidize the focal products.

The overall market size increases when both firms cross-sell. This is obvious when the high type firm decreases its price. Even when it increases its price, the increase is less than the increase in cross-selling surplus for the marginal customer corresponding to the case when neither firm cross-sells. This means that at least one firm always has a higher market size when firms cross-sell compared to when they do not. It is also possible that the market sizes of both firms increase. The formal conditions for the increase in market size of a firm are provided in part (b). Intuitively, the market size of a firm is higher when its cross-selling revenue is sufficiently higher relative to the cross-selling revenue of the competitor.

5.2. Implications of Parameters on Pricing Strategy

In this section, we explore how the equilibrium shifts when one of the exogenous parameters, σ_1, σ_2, β, r_1, and r_2, increases. We discuss what happens to the prices (p_1^* and p_2^*), market sizes (M_1^*, M_2^*, and M^*), and surpluses of customers (denoted by CS_1^* and CS_2^*) when one of the parameters increase.[14]

5.2.1. When Cross-selling Parameter Values Increase We analyze what happens to the price, market size, and surpluses of customers when one of the parameters β, σ_1, or σ_2 increases.

Proposition 3. *(a) When β increases, the price and market size of the high type firm increase, the price and market size of the low type firm decrease, and individual surpluses of all customers increase.*

(b) When σ_1 (σ_2) increases, the prices of both firms decrease, the market size of the low type firm increases (decreases) and that of the high type firm decreases (increases), and individual surpluses of all customers increase.

[14] We provide the analysis of only the pricing game in the main paper and do not treat recommender system effectivenesses as strategic variables because our primary focus is on pricing strategies. For completeness, we also solve the game with recommender system effectivenesses as strategic variables and provide the discussion in Section E of the Appendix.

It follows from the discussion in Section 5.1 that the increase in β increases differentiation and decreases the intensity of the competition. As observed there, the low type firm subsidizes its focal products. The high type firm exploits the decreased competition by increasing its price. Clearly, the surpluses of all individual customers of the low type firm increase. The surpluses of the customers of the high type firm increase because the increase in price is less than the increase in the cross-selling surplus. The results in part (b) are intuitive as the increase in the cross-selling revenue is analogous to a decrease in the cost per unit demand of the corresponding firm in a Bertrand competition – in both cases the profit per unit demand increases. Thus, the results are consistent with what is expected from the existing literature.

5.2.2. The Low Type Firm Improves its Recommender System We next analyze the impact on the price responses, market sizes of firms, and the surpluses of customers when the low type firm improves its recommender system (we assume the increased r_1 remains less than r_2). Customer surpluses change due to (i) changes in their product acquisition costs, and (ii) the changes in the prices of the products from the two firms.

When the low type firm improves its recommender system, the competition between the two firms intensifies because of the decreased differentiation between the firms. The first effect of an increased r_1 is the reduced product acquisition costs from the low type firm. As a result, if neither firm changed its price, some customers of the high type firm (e.g., the indifferent customers) would switch to the low type firm since their surpluses from the low type firm increase. This change in demand prompts the two firms to re-optimize their profits by selecting new prices. Customers also react to the changes in prices and re-optimize their purchase decisions. Based on all this, the customers can be segmented into three groups: (i) retained customers – those who continue purchasing products from the same firm as before, (ii) switchers – those who switch from one firm to the other, and (iii) new customers – customers with $(\theta > \bar{\theta}_2)$ who did not purchase products before and start purchasing products from the high type firm. Proposition 4 summarizes the important changes in the equilibrium (the relevant conditions and expressions are provided in Section A, and the proofs in Section D of the Appendix).

PROPOSITION 4. *When the low type firm improves its recommender system,*

(a) the high type firm always decreases its price regardless of whether the firms cross-sell or not; the low type firm increases its price when $r_1 < r_2\rho_L^{NCS}$ where firms do not cross-sell, and when $r_1 < \tilde{r}_1^L$ where firms cross-sell,

(b) the market size of the high type firm always increases when firms do not cross-sell; when firms cross-sell, the market size decreases if $r_1 > \hat{r}_1^{M_2}$,

(c) the market size of the low type firm always increases when firms do not cross-sell; when firms cross-sell, its market size decreases if $r_1 > \hat{r}_1^{M_1}$.

As stated in part (a), the high type firm always decreases price when the low type firm improves its recommender system. However, the low type firm increases its price only when its effectiveness is less than the threshold values provided in part (a) – $r_2\rho_L^{CS}$ when firms cross-sell and $\tilde{r}_1^L$ when they do not.

⟨Insert Figure 2 here⟩

Figure 2(a) shows the parameter space where p_1^* increases or decreases when the firms cross-sell. It increases in the region where $r_1 < \tilde{r}_1^L$ which is shown in the lower right shaded part of the figure. Conceptually, the differentiation between the firms in this region is very high and the price competition is mild, which allows the low type firm to increase its price. However, when $r_1 > \tilde{r}_1^L$, the price competition becomes intense, forcing the low type firm to decrease its price with any further increase in r_1 (the region shown in the upper shaded portion of the figure). This intuition also applies to the case when firms do not cross-sell, although at a different threshold.

⟨Insert Figure 3 here⟩

The overall market size always increases with r_1 regardless of whether firms cross-sell or not. Interestingly, even though the high type firm reduces its price with an increase in r_1 when $r_1 > \hat{r}_1^{M_2}$ and the firms cross-sell (part (b)), the market size of the high type firm decreases (see Figure 3(b)). Under this condition, the volume of new customers who start to purchase from the high type firm is lower compared to the volume of switchers who switch from the high type firm to the low type firm. This is because the decrease in price by the low type firm is substantially high under this condition. Such a decrease in p_1^* with an increase in r_1 never happens when the firms do not cross-sell – as a result, the market size of the high type firm always increases. Similarly, unlike in the case of no cross-selling, despite improving r_1 the low type firm may end up losing market size (part (c)), as shown in Figure 3(a). When $r_1 > \hat{r}_1^{M_1}$, p_2^* becomes so low that the decrease in p_1^* does not stop customers from migrating to the high type firm, causing a decrease in the market size of the low type firm. In this case, some customers essentially prefer to decrease their search effort by switching to the high type firm rather than using the improved recommender system of the low type firm.

The changes in the profits of both firms when they cross-sell are similar to the changes when they do not cross-sell. The trend in profit of the low type firm mirrors that of its price, i.e., up to a threshold, the profit increases with an increase in r_1. Naturally, the firm would not increase its recommender system beyond this threshold (the condition for the increase in profit is provided in Section A of the Appendix). This is illustrated in Figure 2(b) for the case when firms cross-sell. The profit of the high type firm always decreases; this happens even when its market size increases. Further, all customers have increased surpluses. We find through numerical analysis that the social surplus increases.

5.2.3. The High Type Firm Improves its Recommender System As in the previous case, the equilibrium shifts when the high type firm improves its system. Customers reevaluate their surpluses – some of them may change their associations with the two firms whereas some others who did not purchase earlier may decide to purchase now. These decisions of customers change demands (market sizes) which forces the two firms to reset their prices. Once again, the customers fall into three groups: (i) retained customers, (ii) switchers (who migrate to the other firm), and (iii) new customers. The important insights are presented in Proposition 5 (the relevant expressions and conditions are provided in Section A of the Appendix, and the formal proofs in Section D).

PROPOSITION 5. *When the high type firm improves its recommender system,*

(a) the price of the high type firm always increases regardless of whether the firms cross-sell or not; the price of the low type firm always increases when firms do not cross-sell, but decreases if $r_2 > \frac{r_1}{\rho_H^{CS}}$ *when firms cross-sell,*

(b) the market size of the high type firm always increases when firms do not cross-sell; when they cross-sell, its market size decreases if $r_2 < \hat{r_2}^{M_2}$*,*

(c) the market size of the low type firm always decreases when firms do not cross-sell; when they cross-sell, its market size increases if $r_2 < \hat{r_2}^{M_1}$*.*

When the high type firm improves its recommender system, the differentiation between the firms increases. Intuition from traditional literature on quality differentiation suggests that the prices and profits of both firms should increase, which we observe when the firms do not cross-sell. This result is consistent with events observed in the movie rental business. In 2006, Netflix (the high type firm in this example) publicized its plans to improve its recommender system by instituting a competition with a million-dollar prize (Netflix 2006)[15]. Some months later, Blockbuster (the low type firm) increased its price for its movie rental plans (Hansell 2007, Netflix 2007).[16]

[15]Around 2006, Netflix was already operating a high-quality recommender system called Cinematch, which improved further through the competition entries. Blockbuster, on the other hand, was a new entrant in the online space with a relatively inferior recommender system.

[16]In this context, the two firms provide competing services: the opportunity to watch movies that consumers could rent after paying the monthly subscription fee. Neither firms cross-sold other services or products alongside the subscription services at that time.

However, when firms cross-sell, prices of both firms do not increase always. Although the high type firm increases its price, interestingly, the price of the low type firm increases only when the improved effectiveness is less than a threshold, i.e., $r_2 < \frac{r_1}{\rho_H^{CS}}$. When the new effectiveness $r_2 > \frac{r_1}{\rho_H^{CS}}$, the combined effect of reduced product acquisition cost and increased cross-selling surplus dominates the impact of the increased price of the high type firm to the extent that a large number of customers of the low type firm migrate to the high type firm. To counter, the low type firm decreases its price under this condition.

Surprisingly, unlike the no cross-selling scenario, when the firms cross-sell and the high type firm improves its recommender system, its market size is not guaranteed to increase. With the improved effectiveness of its system, the firm always finds it profitable to increase its price, which would typically lead some of its customers (those close to the indifferent customer) to switch to the low type firm. When the new effectiveness $r_2 < \hat{r}_2^{M_2}$ (part b), because of the increased revenue the high type firm obtains from cross-selling, the firm finds it most profitable to increase its price in such a manner that, even when the low type firm responds by increasing its price, some customers still switch to the low type firm (this never happens when the firms do not cross sell). Further, the volume of switching customers exceeds that of new customers for the high type firm. However, when the effectiveness is higher than the threshold, the increased cross-selling surplus (in addition to the reduced product acquisition costs) the high type firm provides to customers is so high that those close to the indifferent customer end up switching to the high type firm. This occurs even when the low type firm reduces its price in order to abate the migration. The finding about the market size of the low type firm (part c) is similar in nature, although the threshold where the low type firm starts losing its market size is different from that where the high type firm starts gaining its market size. There exists a region where the market sizes of both firms increase.

The reduction in competition intensity allows the firms to adjust their prices in a manner where they are able to extract some of the customer surplus under certain conditions. Thus, some customers end up having reduced surpluses. This occurs both for cross-selling and for no cross-selling scenarios, although with different thresholds of θ. Similar to what was observed when the low type firm improves its system, the improvement in the high type firm's system also leads to some customers switching away from it under certain conditions.

Finally, although the profit of the high type firm always increases as expected, the profit of the low type firm does not necessarily increase with an increase in r_2 (condition provided in Section A of the Appendix), even with the decrease in competition intensity. This never happens when r_2 increases and the firms do not cross-sell. This is also an outcome of the increased cross-selling surplus that the high type firm offers via its improved recommender system, as it limits the amount by which the low type firm can increase its price. The aggregate customer surplus (across both firms) does not increase always (the relevant condition is provided in Section A of the Appendix). We find from numerical experiments that the social surplus always increases.

5.2.4. Summary of Results For ease of exposition, we provide all the results concisely in Table 2. The left column shows the economic variable of our interest. The symbol "↑" ("↓") indicates an increase (decrease) in the variable, and "↑↓" indicates that the variable may either increase or decrease based on the conditions. We highlight the specific results that differ between cross-selling and no cross-selling scenarios in dotted boxes.

⟨Insert Table 2 here.⟩

6. Extension: Some Customers Purchase Without Searching

In this section, we consider the presence of customers in the market who purchase the first recommended product and do not engage in additional examination of products (referred to as *non-strategic customers*) along with the customers who expend effort in searching (i.e., the ones considered so far). A non-strategic customer purchases the first product offered by the firm she visits if the product provides her positive surplus ($R >$ price), or does not purchase anything. Without any loss of generality, we consider that all non-strategic customers purchase a product, and they are γ fraction of the total N customers. The rest of $(1-\gamma)N$ customers are strategic and behave as described in our base model.

6.1. Monopoly

The search cost parameter of the marginal customer remains the same as discussed in Section 4, i.e., $\sqrt{\bar{\theta}} = \frac{r(R-p+\beta r)}{2\sqrt{AB}}$. The profit equation then becomes

$$\Pi = \frac{N(1-\gamma)(p+\sigma)}{2\left(1-\sqrt{\alpha}\right)} \int_{\alpha}^{\bar{\theta}} \frac{1}{\sqrt{\theta}} d\theta + N\gamma p = N(1-\gamma)(p+\sigma) \frac{\left(\sqrt{\bar{\theta}} - \sqrt{\alpha}\right)}{(1-\sqrt{\alpha})} + N\gamma p$$
$$= (1-\gamma)\left(N(p+\sigma)\frac{\left(\sqrt{\bar{\theta}} - \sqrt{\alpha}\right)}{(1-\sqrt{\alpha})} + N\delta p \right),$$

where $\delta = \frac{\gamma}{1-\gamma}$ (using δ, the profit equation can be easily compared with Equation (2)). Ignoring the factor $(1-\gamma)$, the first part of the equation is the revenue from the strategic customers and the second part is the revenue from the non-strategic customers. When this profit function is optimized for the price, we find

$$p^* = \frac{1}{2}\left(R + \beta r - \sigma - \frac{2\sqrt{AB}(\sqrt{\alpha} - \delta(1+\sqrt{\alpha}))}{r} \right).$$

Now, when β or σ increase, we find that p^* increases and decreases, respectively, as observed in the base model. However, when r increases, the optimal price decreases when the fraction of non-strategic customers, γ, exceeds a threshold value.

PROPOSITION 6. *When r increases, the optimal price decreases when $\gamma > \frac{\Omega}{1+\Omega}$, where $\Omega = \left(\frac{\beta r^2}{2\sqrt{AB}} + \sqrt{\alpha}\right)\frac{1}{1-\sqrt{\alpha}}$ when the firm cross-sells, and $\Omega = \frac{\sqrt{\alpha}}{1-\sqrt{\alpha}}$ when the firm does not cross-sell. Otherwise, the optimal price increases.*

This result is an interesting departure from when non-strategic customers do not exist. In that case, we found that the price of the products never decrease with an increase in the recommender system effectiveness. This new result is driven by the new total demand in this case, which depends upon the search cost parameter of the marginal customer. We find that $\bar{\theta}_{\delta>0} < \bar{\theta}_{\delta=0}$. Consequently, the strategic marginal customer is more responsive to the changes in prices and the demand is more elastic (the increase in demand due to decrease in price is higher) when non-strategic customers also exist than when non-strategic customers do not exist. It turns out that when non-strategic customers exist and the proportion of such customers is high, the advantage from decreasing the price dominates the advantage from increasing the price for increasing profit. Referring to our example on e-books in Section 4.1, if there are a large number of non-strategic customers, a virtual monopoly like *Amazon.com* should consider decreasing its prices of e-books if it improves its recommender system.

6.2. Duopoly

We assume that in a duopoly, the demands from these customers are equally distributed between the two firms. The search types of the indifferent customers and the marginal customers is given by Equation (5). The new profit equations are:

$$\Pi_1 = (1-\gamma)\left(N\frac{(\sqrt{\theta_s}-\sqrt{\alpha})}{1-\sqrt{\alpha}}(p_1+\sigma_1)+0.5N\delta p_1\right), \text{ and} \tag{8}$$

$$\Pi_2 = (1-\gamma)\left(N\frac{\left(\sqrt{\bar{\theta}_2}-\sqrt{\theta_s}\right)}{1-\sqrt{\alpha}}(p_2+\sigma_2)+0.5N\delta p_2\right), \tag{9}$$

where $\delta = \frac{\gamma}{1-\gamma}$ in the above equations. We find that the effects of changes in r_1, r_2, β, σ_1, and σ_2 remain the same on prices as in the base model. Even though the prices are adjusted to account for the existence of such customers, they do not cause any qualitative changes in the pricing strategies. This is because the volume of non-strategic customers are equally divided between the two firms, and does not provide any competitive advantage to either firm. For the same reason, no changes are observed in the pricing strategies between when firms cross-sell and when they do not.

7. Conclusions and Future Research

This research advances the existing literature on economics of recommender systems by incorporating two important considerations (i) customer efforts which are dependent on heterogeneous search costs and recommender system effectiveness, and (ii) cross-selling offers made by the retailers with associated cross-selling revenues for firms and additional surplus for customers. We compare two setups – one where both firms do not cross-sell, and another where both firms cross-sell, and find that these two considerations interplay in unique manners that provide several interesting insights.

In a monopoly, even though the firm may charge a higher price when it cross-sells, the customers end up having higher surpluses because the firm extracts only a portion of the surpluses that customers may obtain from cross-selling. When the firm improves its system, then also the customers enjoy higher surpluses from cross-selling even though the firm increases its price. Further, an increase in the expected cross-selling revenue of the firm leads to a decrease in its price.

In a duopoly, firms obtain positive profits by staying vertically differentiated in their recommender system effectivenesses. When both firms cross-sell, competition forces the low type firm to charge a lower price than when they do not cross-sell – a noteworthy departure from what we observe in the monopoly. The high type firm increases its price when the customers' expected cross-selling surplus per unit recommender system effectiveness is more than a threshold. Both firms' market sizes may increase or decrease, depending on the relative values of the cross-selling revenues. The overall market size always increases.

When the low type firm improves its system, it leads to an increase in the intensity of competition. When the original effectiveness of the low type firm's recommender system is relatively high, the firm decreases its price when it improves its recommender system to mitigate the impact of the increased competition intensity. Despite that, when the firms cross-sell, the market size of the low type firm decreases; this never happens when the firms do not cross-sell. Surpluses of all customers increase; however, some customers may switch to the high type firm.

When the high type firm improves its recommender system, the differentiation between the firms increases. When neither firm cross-sells, both firms always increase prices because of reduced competition, the market size of the low type firm decreases, and that of the high type firm increases. In contrast, when firms cross-sell, although the high type firm always increases its price, the low type firm decreases its price when the recommender system effectiveness of the high type firm crosses a threshold. This is because of the additional cross-selling surplus accrued to the customers of the high type firm from the increase in the recommender system effectiveness. Further, the customers of the high type firm with relatively low search cost parameters may experience a decrease in surplus when the high type firm improves its system. Regardless of which firm improves its system, we find that under certain conditions, a few customers may end switching to the other firm. Furthermore, in this process, switchers may end up incurring higher search costs.

Our work provides several opportunities for future research. The assumption of a uniform price for all products in a category limits the types of product categories for which our analysis applies, and can be relaxed. Future research could also relax the assumption of a common reservation prices for all products. Finally, it would be interesting to explore further whether additional insights may result from different distributional assumptions for the search cost parameter.

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8. Figures and Tables

Parameter/ Function	**Description**	**Assumption**
θ	Search cost parameter of a customer	$\theta \in [\alpha, 1]$, $0 < \alpha \leq 1$
r, r_1, r_2	Effectiveness of the recommender system in monopoly and duopoly	$0 \leq \{r, r_1, r_2\}$
R	Reservation price of a customer for her ideal product	Same for the ideal product for each customer
N	Total number of potential customers	
$\psi(\theta)$	pdf for θ	Decreasing and convex in θ
y	Effort exerted by the customer	
β	Cross-selling surplus parameter for customers	$\beta > 0$
p, p_1, p_2	Price in monopoly and duopoly	
A	Positive scaling factor to obtain the utility of search cost for given θ and effort	$A > 0$
B	Positive scaling factor to obtain the utility of mismatch cost for given effort and recommender system effectiveness	$B > 0$
σ, σ_1, σ_2	average revenue per customer who purchases a product in monopoly and duopoly	

Table 1 Parameters and functions used in the model

	β	σ_1	σ_2	r_1		r_2	
				No Cross-selling	Cross-selling	No Cross-selling	Cross-selling
p_1^*	$\downarrow$	$\downarrow$	$\downarrow$	$\uparrow\downarrow$	$\uparrow\downarrow$	$\uparrow$	$\uparrow\downarrow$
p_2^*	$\uparrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\uparrow$	$\uparrow$
M_1^*	$\downarrow$	$\uparrow$	$\downarrow$	$\uparrow$	$\uparrow\downarrow$	$\downarrow$	$\uparrow\downarrow$
M_2^*	$\uparrow$	$\downarrow$	$\uparrow$	$\uparrow$	$\uparrow\downarrow$	$\uparrow$	$\uparrow\downarrow$
Π_1^*	$\downarrow$	$\uparrow$	$\downarrow$	$\uparrow\downarrow$	$\uparrow\downarrow$	$\uparrow$	$\uparrow\downarrow$
Π_2^*	$\uparrow$	$\downarrow$	$\uparrow$	$\downarrow$	$\downarrow$	$\uparrow$	$\uparrow$
M^*	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$
individual CS_1^*	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow\downarrow$
individual CS_2^*	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow\downarrow$	$\uparrow\downarrow$

Table 2 Changes in prices, market sizes, profits, and individual customer surpluses when one of β, σ_1, σ_2, r_1, and r_2 increase. The dotted boxes highlight the results that differ between cross-selling and no cross-selling scenarios.

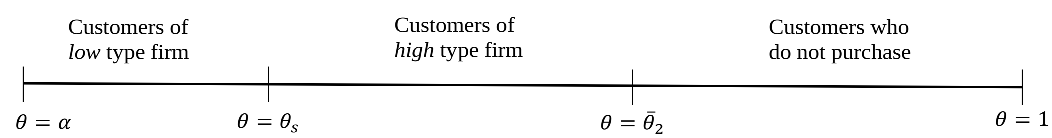

Figure 1 Market with Duopoly

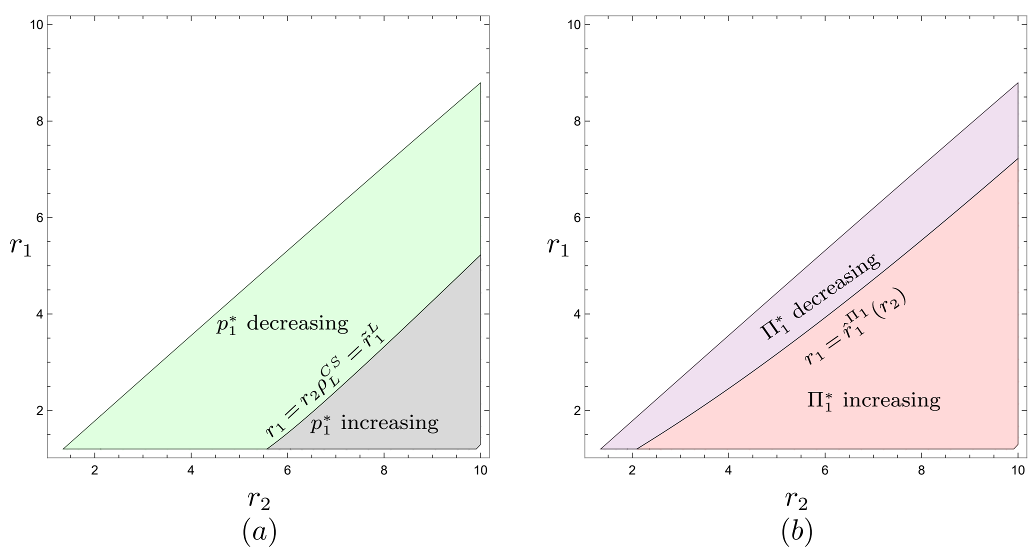

Figure 2 Regions where price and profit of the low type firm increase or decrease when firms cross-sell and r_1 increases ($R=2$, $N=10,000$, $\sqrt{AB}=5$, $\alpha=0.00001$, $\beta=0.2$, $\sigma_1=0.04391$, and $\sigma_2=0.124$). The expression in panel (a) represents the contour of r_1 that separates the two regions. The expression in panel (b) represents the root of $\phi_2(r_1,r_2)=0$ (Section A of the Appendix).

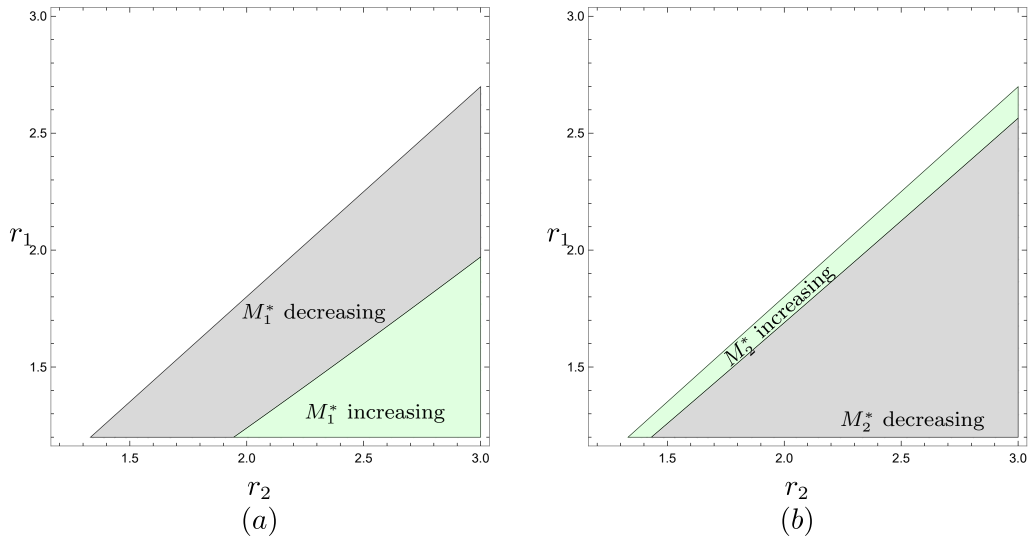

Figure 3 Regions where market sizes of the low type and high type firms increase and decrease when firms cross-sell and r_2 increases ($R = 2$, $N = 10,000$, $\sqrt{AB} = 5$, $\alpha = 0.00001$, $\beta = 0.2$, $\sigma_1 = 0.01508$, and $\sigma_2 = 0.26875$)

Appendix.

This appendix contains important expressions used in the paper in Section A, derivation for the monopoly price in Section B, duopoly price equilibrium and results in Section C, proofs of lemmas and propositions in Section D, equilibrium analysis for the recommender system effectivenesses in Section E, and the analysis of the case when firms cross-sell products that are same as the focal products in Section F.

A. Important Expressions:

A.1. Conditions related to Proposition 4

$$\rho_L^{NCS} = \frac{4\sqrt{\alpha}\sqrt{AB}(-1+\sqrt{3(R-\sqrt{\alpha}\sqrt{AB})})}{3Rr_2 - 4\sqrt{\alpha}\sqrt{AB}}.$$

The $\tilde{r}_1^L$ is a root of the following equation:

$$\beta r_1^4 - 8r_2\beta r_1^3 + \left(-3Rr_2 + 4\sqrt{AB\alpha} + r_2(7r_2\beta - 2\sigma_1 - \sigma_2)\right) r_1^2 - 8r_1r_2\sqrt{AB\alpha} + 16r_2^2\sqrt{AB\alpha} = 0.$$

Condition for Π_1^* to increase with an increase in r_1 when firms cross-sell

$$\begin{aligned}\phi_2(r_1) &= -2R\left(r_1 - r_2\right) r_2 \left(r_1 + 2r_2\right) + \left(2r_1^2 + 2r_1r_2 - 4r_2^2\right) \sqrt{AB}\sqrt{\alpha} \\ &+ \left(r_1^3 r_2 - 15r_1^2 r_2^2 + 18r_1 r_2^3 - 4r_2^4\right)\beta + \left(-r_1^2 r_2 - 4r_1 r_2^2 + 8r_2^3\right)\sigma_1 + \left(-r_1^2 r_2 + 2r_1 r_2^2 - 4r_2^3\right)\sigma_2 > 0\end{aligned}$$

Condition for Π_1^* to increase with an increase in r_1 when firms do not cross-sell

$$\frac{2\left(Rr_2^2 - 3r_2\sqrt{AB\alpha} + r_2\sqrt{\left(Rr_2 - \sqrt{AB\alpha}\right)\left(Rr_2 + 23\sqrt{AB\alpha}\right)}\right)}{7Rr_2 - 8\sqrt{AB\alpha}} > 0.$$

A.2. Conditions related to Proposition 5

$$\rho_H^{CS} = \frac{-\left(3R+2\beta r_2+2\sigma_1+\sigma_2\right)+\sqrt{\left(3R+2\beta r_2+2\sigma_1+\sigma_2\right)^2+32\beta\left(\beta r_2^2+3\sqrt{\alpha}\sqrt{AB}\right)}}{4\beta r_2}$$

When the high type firm increases r_2, customers with $\theta > \theta_{ls}^{CS}$ experience decreased surplus when the firms cross-sell, and customers with $\theta > \theta_{ls}^{NCS}$ experience decreased surplus when the firms do not cross-sell.

$$\theta_{ls}^{CS} = \frac{3Rr_1r_2^2+r_2^2(r_1(2\sigma_1+\sigma_2)-2\beta\left(r_2-r_1\right)\left(r_1+2r_2\right))+\sqrt{\alpha}\sqrt{AB}\left(r_1^2-8r_1r_2+4r_2^2\right)}{\sqrt{AB}\left(4r_2-r_1\right)^2},$$

$$\theta_{ls}^{NCS} = \frac{3Rr_1r_2^2+\sqrt{\alpha}\sqrt{AB}\left(r_1^2-8r_1r_2+4r_2^2\right)}{\sqrt{AB}\left(4r_2-r_1\right)^2}.$$

Condition for Π_1^* to increase with an increase in r_2 when both firms cross-sell

$$\left(\begin{array}{c} 2R\left(r_1-r_2\right)\left(2r_1^2-3r_1r_2+4r_2^2\right)+\sqrt{AB}\sqrt{\alpha}\left(-14r_1^2+22r_1r_2-8r_2^2\right) \\ +\beta\left(2r_1^4+3r_1^3r_2-27r_1^2r_2^2+46r_1r_2^3-24r_2^4\right)+\left(2r_1^3-r_1^2r_2-4r_1r_2^2\right)\sigma_1+ \\ \left(2r_1^3-9r_1^2r_2+18r_1r_2^2-8r_2^3\right)\sigma_2 \end{array}\right)<0$$

The condition $\Phi_H > 0$ should hold for the total surplus to increase, where

$$\Phi_H = \frac{d\Sigma_1^*}{dr_2}+\frac{d\Sigma_2^*}{dr_2}. \tag{10}$$

and the aggregate surpluses corresponding to the low type and the high type firms, respectively, are the following:

$$\begin{aligned}\Sigma_1^* = & \frac{1}{2r_1\left(r_1-4r_2\right)^2\left(r_1-r_2\right)^2\sqrt{AB}}\Big(r_2\Big(5Rr_1\left(r_1-r_2\right)r_2+4r_2^2\sqrt{AB}\sqrt{\alpha}+3r_1^3r_2\beta \\ & -r_1r_2^2\left(3r_2\beta+2\sigma_1+3\sigma_2\right)+r_1^2\left(-4\sqrt{AB}\sqrt{\alpha}+3r_2\sigma_1+2r_2\sigma_2\right)\Big)\Big(Rr_1\left(r_1-r_2\right)+4r_2\sqrt{AB}\sqrt{\alpha} \\ & +r_1^3\beta+r_1^2\left(-2r_2\beta+\sigma_1\right)+r_1\left(-4\sqrt{AB}\sqrt{\alpha}+r_2\left(r_2\beta-2\sigma_1+\sigma_2\right)\right)\Big)\Big), \text{ and}\end{aligned} \tag{11}$$

$$\begin{aligned}\Sigma_2^* = & \frac{1}{2\left(r_1-4r_2\right)^2\left(r_1-r_2\right)^2\sqrt{AB}}\Big(r_2\Big(2R\left(r_1-r_2\right)r_2+r_1^2r_2\beta-2r_2 \\ & \left(-\sqrt{AB}\sqrt{\alpha}+r_2\left(r_2\beta+\sigma_2\right)\right)+r_1\left(-2\sqrt{AB}\sqrt{\alpha}+r_2\left(r_2\beta+\sigma_1+\sigma_2\right)\right)\Big)^2\Big).\end{aligned} \tag{12}$$

B. Monopoly Price Decision

When the firm cross-sells:

$$S = R - A\theta y - \frac{B}{r^2 y} - p + \beta r.$$

By differentiating S with respect to y we get,

$$\frac{dS}{dy} = -A\theta + \frac{B}{r^2 y^2}; \qquad \frac{d^2 S}{dy^2} = \frac{-2B}{r^2 y^3} < 0.$$

From $\frac{dS}{dy} = 0$ we get $y^* = \frac{\sqrt{B}}{\sqrt{A\theta r^2}}$. The second order implies that this effort is surplus maximizing. Substituting that in S, we get $S^* = R - \frac{2\sqrt{AB\theta}}{r} - p + \beta r$. Using $S^* = 0$, we can find the search type of the marginal customer, which is $\sqrt{\bar{\theta}} = \frac{r(R-p+\beta r)}{2\sqrt{AB}}$. The demand, or the market size, is $M = \frac{N}{(1-\sqrt{\alpha})}\left(\sqrt{\bar{\theta}} - \sqrt{\alpha}\right)$, and the profit of the firm is $\Pi = N\frac{p+\sigma}{(1-\sqrt{\alpha})}\left(\sqrt{\bar{\theta}} - \sqrt{\alpha}\right)$. Now,

$$\frac{d\Pi}{dp} = \frac{N}{1-\sqrt{\alpha}} \frac{r(R+\beta r - 2p - \sigma) - 2\sqrt{AB\alpha}}{2\sqrt{AB}}, \quad \frac{d^2\Pi}{dp^2} = \frac{-Nr}{\sqrt{AB}(1-\sqrt{\alpha})} < 0.$$

Using first order condition we derive optimal price (p^*). From the second order condition, we know that p^* maximizes profit. Through p^*, we get equilibrium $\sqrt{\bar{\theta}}$, optimal profit, market size, and individual customer surplus as

$$p^* = \frac{1}{2}\left(R - \frac{2\sqrt{AB}\sqrt{\alpha}}{r} + \beta r - \sigma\right),\ \Pi^* = \frac{N\left(-2\sqrt{AB}\sqrt{\alpha} + r(R + r\beta + \sigma)\right)^2}{8\sqrt{AB}r\,(1-\sqrt{\alpha})}, \tag{13}$$

$$M^* = \frac{N\left(r(R + r\beta + \sigma) - 2\sqrt{AB}\sqrt{\alpha}\right)}{4\sqrt{AB}\,(1-\sqrt{\alpha})},\ CS^* = \frac{1}{2}\left(R - \frac{2\sqrt{AB}\sqrt{\theta}}{r} + \beta r + \sigma\right). \tag{14}$$

When the firm does not cross-sell:

$$S = R - A\theta y - \frac{B}{r^2 y} - p.$$

As in the previous case,

$$\frac{dS}{dy} = -A\theta + \frac{B}{r^2 y^2}; \qquad \frac{d^2 S}{dy^2} = \frac{-2B}{r^2 y^3} < 0.$$

We can determine the y^* from the first order derivative, and the resulting S^* is a maxima as it is evident from the sign of the second order derivative. We can also now derive the value of $\bar{\theta}$ when the firm does not cross-sell.

Next, we determine the demand of the products of the firm. Finally, we find the optimal price, profit, market size, and individual customer surplus. Now, the profit of the firm is $\Pi = N\frac{p}{(1-\sqrt{\alpha})}\left(\sqrt{\theta} - \sqrt{\alpha}\right)$, where $\sqrt{\theta} = \frac{r(R-p)}{2\sqrt{AB}}$. Therefore,

$$\frac{d\Pi}{dp} = \frac{N}{1-\sqrt{\alpha}}\frac{r(R-2p) - 2\sqrt{AB\alpha}}{2\sqrt{AB}}, \quad \frac{d^2\Pi}{dp^2} = \frac{-Nr}{\sqrt{AB}(1-\sqrt{\alpha})} < 0.$$

The optimal price p^* for this case can be determined by equating the first order derivative to zero and solving for p. The resulting price is profit maximizing which is implied from the second order condition.

$$p^* = \frac{1}{2}\left(R - \frac{2\sqrt{AB}\sqrt{\alpha}}{r}\right), \Pi^* = \frac{N\left(-2\sqrt{AB}\sqrt{\alpha} + Rr\right)^2}{8\sqrt{AB}r\,(1-\sqrt{\alpha})}, \tag{15}$$

$$M^* = \frac{N\left(Rr - 2\sqrt{AB}\sqrt{\alpha}\right)}{4\sqrt{AB}\,(1-\sqrt{\alpha})}, CS^* = \frac{1}{2}\left(R - \frac{2\sqrt{AB}\sqrt{\theta}}{r}\right). \tag{16}$$

B.1. Boundary condition in Equation (3)

When firms cross-sell, since $\sqrt{\theta} = \left(\frac{r(R+\beta r+\sigma)+2\sqrt{AB\alpha}}{4\sqrt{AB}}\right)$ and we know that $\sqrt{\alpha} < \sqrt{\theta}$, we get the upper bound on $\sqrt{AB}$ using this condition. Likewise, using $\sqrt{\theta} < 1$, we get the lower bound on $\sqrt{AB}$. The bounds on $\sqrt{AB}$ for the case when the firm does not cross-sell can be found using the same process.

C. Simultaneous Move Price Equilibrium

When firms cross-sell: A customer purchases a product from that firm which provides her a higher expected surplus. The firms choose prices simultaneously for given r_1 and r_2 in order to maximize their profits. The profits of the low type firm and the high type firm, respectively, are

$$\Pi_1 = N\frac{(\sqrt{\theta_s} - \sqrt{\alpha})}{1-\sqrt{\alpha}}(p_1 + \sigma_1), \text{ and} \tag{17}$$

$$\Pi_2 = N\frac{\left(\sqrt{\theta_2} - \sqrt{\theta_s}\right)}{1-\sqrt{\alpha}}(p_2 + \sigma_2). \tag{18}$$

Differentiating the above profit equations with respect to p_1 and p_2, respectively,

$$\frac{\partial \Pi_1}{\partial p_1} = \frac{N\left(-2p_1r_1r_2 + p_2r_1r_2 - (r_2-r_1)\left(2\sqrt{AB}\sqrt{\alpha} + r_1r_2\beta\right) - r_1r_2\sigma_1\right)}{2(r_2-r_1)\sqrt{AB}\left(1-\sqrt{\alpha}\right)} \quad \frac{\partial^2\Pi_1}{\partial p_1^2} = \frac{-Nr_1r_2}{(r_2-r_1)\sqrt{AB}\left(1-\sqrt{\alpha}\right)} < 0$$

$$\frac{\partial \Pi_2}{\partial p_2} = \frac{Nr_2(p_1r_1 - 2p_2r_2 + (r_2-r_1)(R + (r_1+r_2)\beta) - r_2\sigma_2)}{2(r_2-r_1)\sqrt{AB}\left(1-\sqrt{\alpha}\right)} \quad \frac{\partial^2\Pi_2}{\partial p_2^2} = \frac{-Nr_2^2}{(r_2-r_1)\sqrt{AB}\left(1-\sqrt{\alpha}\right)} < 0.$$

The above first order equations are simultaneously solved to determine the equilibrium prices. From the second order conditions, it is clear that the prices maximize profits in equilibrium. Using these prices, we find the equilibrium market sizes and the profits of the firms.

CLAIM 1. *(a) The equilibrium profits of the low type and the high type firms, respectively, are*

$$\Pi_1^* = \frac{Nr_2\left(Rr_1(r_1-r_2)+4\sqrt{A}\sqrt{B}(-r_1+r_2)\sqrt{\alpha}+r_1\left(r_1^2\beta+r_1(-2r_2\beta+\sigma_1)+r_2(r_2\beta-2\sigma_1+\sigma_2)\right)\right)^2}{2\sqrt{A}\sqrt{B}r_1(4r_2-r_1)^2(r_2-r_1)(1-\sqrt{\alpha})};$$

$$\Pi_2^* = \frac{N\left(2R(r_1-r_2)r_2+2\sqrt{A}\sqrt{B}(-r_1+r_2)\sqrt{\alpha}+r_2\left(r_1^2\beta-2r_2(r_2\beta+\sigma_2)+r_1(r_2\beta+\sigma_1+\sigma_2)\right)\right)^2}{2\sqrt{A}\sqrt{B}(4r_2-r_1)^2(r_2-r_1)(1-\sqrt{\alpha})}.$$

(b) The equilibrium prices of the low type and the high type firms, respectively, are

$$p_1^* = \frac{(r_2-r_1)\left(Rr_1-4\sqrt{A}\sqrt{B}\sqrt{\alpha}+r_1(r_1-r_2)\beta\right)}{r_1(4r_2-r_1)} - \frac{r_2(2\sigma_1+\sigma_2)}{4r_2-r_1}; \tag{19}$$

$$p_2^* = \frac{(r_2-r_1)\left(2Rr_2-2\sqrt{A}\sqrt{B}\sqrt{\alpha}+r_2(r_1+2r_2)\beta\right)}{(4r_2-r_1)r_2} - \frac{r_1\sigma_1+2r_2\sigma_2}{4r_2-r_1}. \tag{20}$$

(c)The equilibrium market sizes of the low type and the high type firms, respectively, are

$$M_1^* = \frac{Nr_2\left(Rr_1-4\sqrt{A}\sqrt{B}\sqrt{\alpha}+r_1(r_1-r_2)\beta\right)}{2\sqrt{A}\sqrt{B}(4r_2-r_1)(1-\sqrt{\alpha})} + \frac{Nr_1r_2(-r_1\sigma_1+r_2(2\sigma_1-\sigma_2))}{2\sqrt{A}\sqrt{B}(4r_2-r_1)(1-\sqrt{\alpha})(r_2-r_1)}; \tag{21}$$

$$M_2^* = \frac{Nr_2\left(2Rr_2-2\sqrt{A}\sqrt{B}\sqrt{\alpha}+r_2(r_1+2r_2)\beta\right)}{2\sqrt{A}\sqrt{B}(4r_2-r_1)(1-\sqrt{\alpha})} + \frac{Nr_2^2(2r_2\sigma_2-r_1(\sigma_1+\sigma_2))}{2\sqrt{A}\sqrt{B}(4r_2-r_1)(1-\sqrt{\alpha})(r_2-r_1)} \tag{22}$$

In the equilibrium, the θ's of the indifferent customer and the marginal customer, respectively, are

$$\theta_s = \frac{1}{(2\sqrt{A}\sqrt{B}(4r_2-r_1)(r_2-r_1))^2}\Big(Rr_1r_2(-r_1+r_2)+2\sqrt{A}\sqrt{B}(r_1-2r_2)(r_1-r_2)\sqrt{\alpha}-r_1r_2$$
$$\left(r_1^2\beta+r_1(-2r_2\beta+\sigma_1)+r_2(r_2\beta-2\sigma_1+\sigma_2)\right)\Big)^2 \text{ and} \tag{23}$$

$$\bar{\theta}_2 = \left(\frac{Rr_2(r_1+2r_2)+2\sqrt{A}\sqrt{B}(-r_1+r_2)\sqrt{\alpha}+r_2\left(r_1^2\beta+r_1\sigma_1+2r_2(r_2\beta+\sigma_2)\right)}{2\sqrt{A}\sqrt{B}(4r_2-r_1)}\right)^2. \tag{24}$$

<u>When firms do not cross-sell</u>: Likewise, we can derive the profits, prices, and market sizes of the firms when they do not cross-sell. The first and second order conditions are the following.

$$\frac{\partial\Pi_1}{\partial p_1} = \frac{N\left(-2\sqrt{AB}\sqrt{\alpha}(r_2-r_1)-2p_1r_1r_2+p_2r_1r_2\right)}{2\sqrt{AB}\left(1-\sqrt{\alpha}\right)(r_2-r_1)} \quad \frac{\partial^2\Pi_1}{\partial p_1^2} = \frac{-Nr_1r_2}{\sqrt{AB}\left(1-\sqrt{\alpha}\right)(r_2-r_1)}$$

$$\frac{\partial\Pi_2}{\partial p_2} = \frac{Nr_2(p_1r_1+R(r_2-r_1)-2p_2r_2)}{2\sqrt{AB}\left(1-\sqrt{\alpha}\right)(r_2-r_1)} \quad \frac{\partial^2\Pi_2}{\partial p_2^2} = \frac{-Nr_2^2}{\sqrt{AB}\left(1-\sqrt{\alpha}\right)(r_2-r_1)}$$

By solving the first order conditions simultaneously, we determine the prices that maximize profits in the equilibrium (implied from the second order conditions), and then the profits and market sizes in equilibrium.

CLAIM 2. *(a) The equilibrium profits of the low type and the high type firms, respectively, are*

$$\Pi_1^* = \frac{N(r_2 - r_1)r_2\left(Rr_1 - 4\sqrt{AB\alpha}\right)^2}{2\sqrt{AB}r_1\left(4r_2 - r_1\right)^2\left(1 - \sqrt{\alpha}\right)} \text{ and}$$
$$\Pi_2^* = \frac{N(r_2 - r_1)\left(2Rr_2 - 2\sqrt{AB\alpha}\right)^2}{2\sqrt{AB}\left(4r_2 - r_1\right)^2\left(1 - \sqrt{\alpha}\right)}$$

(b) The equilibrium prices of the low type and the high type firms, respectively, are

$$p_1^* = \frac{\left(r_2 - r_1\right)\left(Rr_1 - 4\sqrt{A}\sqrt{B}\sqrt{\alpha}\right)}{r_1\left(4r_2 - r_1\right)} \text{ and } p_2^* = \frac{\left(r_2 - r_1\right)\left(2Rr_2 - 2\sqrt{A}\sqrt{B}\sqrt{\alpha}\right)}{\left(4r_2 - r_1\right)r_2}.$$

(c)The equilibrium market sizes of the low type and the high type firms, respectively, are

$$M_1^* = \frac{Nr_2\left(Rr_1 - 4\sqrt{A}\sqrt{B}\sqrt{\alpha}\right)}{2\sqrt{A}\sqrt{B}\left(4r_2 - r_1\right)\left(1 - \sqrt{\alpha}\right)} \text{ and } M_2^* = \frac{Nr_2\left(2Rr_2 - 2\sqrt{A}\sqrt{B}\sqrt{\alpha}\right)}{2\sqrt{A}\sqrt{B}\left(4r_2 - r_1\right)\left(1 - \sqrt{\alpha}\right)}.$$

In the equilibrium, the θ's of the indifferent customer and the marginal customer, respectively, are

$$\theta_s = \left(\frac{Rr_1r_2 + 2\sqrt{A}\sqrt{B}\left(2r_2 - r_1\right)\sqrt{\alpha}}{2\sqrt{A}\sqrt{B}\left(4r_2 - r_1\right)}\right)^2 \text{ and } \bar{\theta}_2 = \left(\frac{Rr_2\left(r_1 + 2r_2\right) + 2\sqrt{A}\sqrt{B}\left(r_2 - r_1\right)\sqrt{\alpha}}{2\sqrt{A}\sqrt{B}\left(4r_2 - r_1\right)}\right)^2. \tag{25}$$

C.1. Deriving the boundary condition in Equation (6)

When firms cross-sell, by using $\sqrt{\bar{\theta}_2} < 1$, we obtain the lower bound of this equation, and by using $\sqrt{\alpha} < \sqrt{\theta_s}$, we can obtain the upper bound on $\sqrt{AB}$. The boundary conditions for the case when firms do not cross-sell can be derived in the similar way.

D. Proofs of Propositions and Lemmas

Proof of Proposition 1: By comparing the prices in Equations (13) and (15), and market sizes and surpluses of individual customers in Equations (14) and (16), we can obtain the results stated in the proposition.

We also provide the proofs for the following:

When the firm cross-sells and:

(a) the customer's expected cross-selling surplus (parameter β) increases, the price, market size, and customer surplus increase.

$\frac{dp^*}{d\beta} = \frac{r}{2} > 0$, $\frac{dM^*}{d\beta} = \frac{Nr^2}{4\sqrt{AB}(1-\sqrt{\alpha})} > 0$, and $\frac{dCS^*}{d\beta} = \frac{r}{2} > 0$.

(b) the firm's expected cross-selling revenue (parameter σ) increases, price decreases, market size, and customer surplus increases.

$\frac{dp^*}{d\sigma} = \frac{-1}{2} < 0$, $\frac{dM^*}{d\sigma} = \frac{Nr}{4\sqrt{AB}(1-\sqrt{\alpha})} > 0$, and $\frac{dCS^*}{d\sigma} = \frac{1}{2} > 0$.

(c) it improves its recommender system, the price, the market size, as well as the aggregate customer surplus, increase.

$\frac{dp^*}{dr} = \frac{\sqrt{AB\alpha}}{r^2} + \frac{\beta}{2} > 0$, $\frac{dM^*}{dr} = \frac{N(R+2r\beta+\sigma)}{4\sqrt{AB}\left(1-\sqrt{\alpha}\right)} > 0$, and $\frac{dCS^*}{dr} = \frac{1}{2}\left(\frac{2\sqrt{AB}\left(2\sqrt{\theta}-\sqrt{\alpha}\right)}{r^2} + \beta\right) > 0$.

Proof of Lemma 1: Using Equation (4), we find that

$$S_2^* - S_1^* = \Delta S^* = 2\sqrt{AB}\sqrt{\theta}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) - (p_2 - p_1) + \beta\,(r_2 - r_1)\,. \tag{26}$$

Considering customers with $\max(S_1^*, S_2^*) \geq 0$ (since only these customers purchase products), we use the condition $\Delta S^* = 0$ to determine the search cost parameter of the customer with $\theta = \theta_s$; the condition yields

$$\sqrt{\theta_s} = \frac{r_1 r_2\,(p_2 - p_1 - \beta\,(r_2 - r_1))}{2\sqrt{AB}\,(r_2 - r_1)} \implies \theta_s = \left(\frac{r_1 r_2\,(p_2 - p_1 - \beta\,(r_2 - r_1))}{2\sqrt{AB}\,(r_2 - r_1)}\right)^2.$$

Since $\sqrt{\theta_s} > 0$ and $r_2 - r_1 > 0$, therefore,

$$p_2 - p_1 - \beta\,(r_2 - r_1) > 0 \implies p_2 - p_1 > 0.$$

Now, considering a market where both firms sell products to customers. We know that a customer purchases from the low type firm if $\Delta S^* \leq 0$ for her and $S_1^* \geq 0$. Thus, for such a customer,

$$2\sqrt{AB}\sqrt{\theta}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) - (p_2 - p_1) + \beta\,(r_2 - r_1) \leq 0,$$
$$\implies \sqrt{\theta} \leq \frac{r_1 r_2\,(p_2 - p_1 - \beta\,(r_2 - r_1))}{2\sqrt{AB}\,(r_2 - r_1)} = \sqrt{\theta_s}.$$

The indifferent customer ($\theta = \theta_s$) receives positive surplus as otherwise the high type firm would have no customers. It follows from Equation (4) that $S_1^*(\theta) > S_1^*(\theta_s)$ when $\theta < \theta_s$, and therefore, $S_1^*(\theta) > 0$. Hence, customers with search cost parameters $\theta < \theta_s$ purchase products from the low type firm.

Likewise, the customer purchases from the high type firm if $\Delta S^* > 0$ for her and $S_2^* \geq 0$. Thus, for such a customer,

$$2\sqrt{AB}\sqrt{\theta}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) - (p_2 - p_1) + \beta(r_2 - r_1) > 0 \implies \sqrt{\theta} > \sqrt{\theta_s}.$$

Therefore, customers with search cost parameter $\theta > \theta_s$ purchase products from the high type firm. Next, we show that the marginal customer ($\theta = \bar{\theta}$) purchases products from the high type firm. We know that the indifferent customer with $\theta = \theta_s$ purchases products (as both firms sell products in the setup we consider) which implies that her surplus $S^*(\theta_s) > 0$. Also, from Equation (4), we know that the surplus obtained from a firm monotonically decreases with an increase in θ (this is true regardless of which firm the customer purchases products from). Since by definition the marginal customer with $\theta = \bar{\theta}$ has surplus $S^* = 0$, therefore, $\theta_s < \bar{\theta}$. Hence, customers with $\theta \epsilon [\alpha, \theta_s]$ purchase products from the low type firm and customers with $\theta \epsilon (\theta_s, \bar{\theta}]$ purchase products from the high type firm.

Proof of Proposition 2: Difference between p_1^* when firms cross-sell and when firms do not cross-sell is $\Delta p_1^* = \frac{-\beta r_1^2 + 2\beta r_1 r_2 - r_2(\beta r_2 + 2\sigma_1 + \sigma_2)}{4r_2 - r_1} < 0$. Difference between p_2^* when firms cross-sell and when firms do not cross-sell is $\Delta p_2^* = \frac{-\beta r_1^2 - r_1(\beta r_2 + \sigma_1) - 2r_2(-\beta r_2 + \sigma_2)}{4r_2 - r_1}$. The condition for the increase in p_2^* can be determined by solving $\Delta p_2^* > 0$. The other conditions can be derived by comparing the market sizes when firms cross-sell with when they do not cross-sell.

Proof of Proposition 3: Note that $r_2 > r_1$. Increase in β when firms cross-sell:

$$\frac{dp_1^*}{d\beta} = \frac{-(r_2 - r_1)^2}{4r_2 - r_1} < 0, \ \frac{dp_2^*}{d\beta} = \frac{(r_2 - r_1)(r_1 + 2r_2)}{4r_2 - r_1} > 0,$$

$$\frac{dM_1^*}{d\beta} = \frac{-Nr_1(r_2 - r_1)r_2}{2\sqrt{AB}(1 - \sqrt{\alpha})(4r_2 - r_1)} < 0, \ \frac{dM_2^*}{d\beta} = \frac{Nr_2^2(r_1 + 2r_2)}{2\sqrt{AB}(1 - \sqrt{\alpha})(4r_2 - r_1)} > 0,$$

Following the above deductions, we find that Π_1^* decreases and Π_2^* increases when β increases. For individual consumer surpluses, derivatives with respect to β are

$$\frac{CS_1^*}{d\beta} = \frac{r_2\,(2r_1+r_2)}{4r_2-r_1} > 0 \text{ and, } \frac{CS_2^*}{d\beta} = \frac{r_1^2+2r_2^2}{4r_2-r_1} > 0.$$

Increase in σ_1:

$$\frac{dp_1^*}{d\sigma_1} = \frac{-2r_2}{4r_2-r_1} < 0,\ \frac{dp_2^*}{d\sigma_1} = \frac{-r_1}{4r_2-r_1} < 0,$$

$$\frac{dM_1^*}{d\sigma_1} = \frac{Nr_1\,(2r_2-r_1)\,r_2}{2\sqrt{AB}\,(1-\sqrt{\alpha})\,(4r_2-r_1)\,(r_2-r_1)} > 0,\ \frac{dM_2^*}{d\sigma_1} = \frac{-Nr_1r_2^2}{2\sqrt{AB}\,(1-\sqrt{\alpha})\,(r_2-r_1)(4r_2-r_1)} < 0,$$

$$\frac{dCS_1^*}{d\sigma_1} = \frac{2r_2}{4r_2-r_1} > 0, \text{ and } \frac{dCS_2^*}{d\sigma_1} = \frac{r_1}{4r_2-r_1}.$$

Suppose,

$\phi_1 = (2Nr_2(Rr_1\,(r_2-r_1) - 4\,(-r_1+r_2)\,\sqrt{AB}\sqrt{\alpha} - (r_1^3 - 2r_1^2r_2 + r_1r_2^2)\,\beta + (-r_1^2+2r_1r_2)\,\sigma_1 - r_1r_2\sigma_2)).$

Now, $\phi_1 > 0$ follows from $M_1^* > 0$. Therefore,

$$\frac{d\Pi_1^*}{d\sigma_1} = \frac{\phi_1}{2\sqrt{AB}r_1\,(r_1-4r_2)^2\,(r_2-r_1)\,(1-\sqrt{\alpha})}(2r_1r_2-r_1^2) > 0.$$

Suppose,

$\phi_2 = (2N(2R\,(r_2-r_1)\,r_2 - 2\,(r_2-r_1)\,\sqrt{AB}\sqrt{\alpha} - (r_1^2r_2 + r_1r_2^2 - 2r_2^3)\,\beta - r_1r_2\sigma_1 + (-r_1r_2+2r_2^2)\,\sigma_2).$

Now, $\phi_2 > 0$ since $M_2^* > 0$. Therefore,

$$\frac{d\Pi_2^*}{d\sigma_1} = \frac{\phi_2}{2\sqrt{AB}\,(r_1-4r_2)^2\,(r_2-r_1)\,(1-\sqrt{\alpha})}(-r_1r_2) < 0.$$

Increase in σ_2:

$$\frac{dp_1^*}{d\sigma_2} = \frac{-r_2}{4r_2-r_1} < 0,\ \frac{dp_2^*}{d\sigma_2} = \frac{-2r_2}{4r_2-r_1} < 0,$$

$$\frac{dM_1^*}{d\sigma_2} = \frac{-Nr_1r_2^2}{2\sqrt{AB}\,(1-\sqrt{\alpha})\,(4r_2-r_1)(r_2-r_1)} < 0,\ \frac{dM_2^*}{d\sigma_2} = \frac{N\,(2r_2-r_1)\,r_2^2}{2\sqrt{AB}\,(1-\sqrt{\alpha})\,(4r_2-r_1)(r_2-r_1)} > 0.$$

Suppose,

$\phi_1 = (2Nr_2(Rr_1\,(r_2-r_1) - 4\,(-r_1+r_2)\,\sqrt{AB}\sqrt{\alpha} - (r_1^3 - 2r_1^2r_2 + r_1r_2^2)\,\beta + (-r_1^2+2r_1r_2)\,\sigma_1 - r_1r_2\sigma_2)).$

Now, $\phi_1 > 0$ follows from $M_1^* > 0$. Therefore,

$$\frac{d\Pi_1^*}{d\sigma_2} = \frac{\phi_1}{2\sqrt{AB}r_1\,(r_1-4r_2)^2\,(r_2-r_1)\,(1-\sqrt{\alpha})}(-r_1r_2) < 0.$$

Suppose,

$\phi_2 = (2N(2R\,(r_2 - r_1)\,r_2 - 2\,(r_2 - r_1)\,\sqrt{AB}\sqrt{\alpha} - (r_1^2 r_2 + r_1 r_2^2 - 2r_2^3)\,\beta - r_1 r_2 \sigma_1 + (-r_1 r_2 + 2r_2^2)\,\sigma_2)$.

Now, $\phi_2 > 0$ follows from $M_2^* > 0$. Therefore,

$$\frac{d\Pi_2^*}{d\sigma_2} = \frac{\phi_2}{2\sqrt{AB}\,(r_1 - 4r_2)^{2}\,(r_2 - r_1)\,(1 - \sqrt{\alpha})}(2r_2^2 - r_1 r_2) > 0.$$

$$\frac{dCS_1^*}{d\sigma_2} = \frac{r_2}{4r_2 - r_1} > 0,\ \frac{dCS_2^*}{d\sigma_2} = \frac{2r_2}{4r_2 - r_1} > 0.$$

Proof of Proposition 4: We differentiate equilibrium prices and market sizes of the firms, and customer surplus with respect to r_1 in order to find the changes in these when the low type firm improves its recommender system.

<u>When firms do not cross-sell:</u>

(a) $\frac{dp_2^*}{dr_1} = \frac{6\sqrt{AB}\sqrt{\alpha} - 6Rr_2}{(r_1 - 4r_2)^2} < 0$ and the condition for the increase in p_1^* can be obtained by using $\frac{dp_1^*}{dr_1} > 0$.

(b,c) Since $M_1^* > 0$, we have $\sqrt{AB}\sqrt{\alpha} < \frac{Rr_1}{4}$. Using this we can easily show the following:

$$\frac{dM_1^*}{dr_1} = \frac{2Nr_2\left(\sqrt{A}\sqrt{B}\sqrt{\alpha} - Rr_2\right)}{\sqrt{A}\sqrt{B}\,(-1 + \sqrt{\alpha})\,(r_1 - 4r_2)^{2}} > 0,\ \frac{dM_2^*}{dr_1} = \frac{Nr_2\left(\sqrt{A}\sqrt{B}\sqrt{\alpha} - Rr_2\right)}{\sqrt{A}\sqrt{B}\,(-1 + \sqrt{\alpha})\,(r_1 - 4r_2)^{2}} > 0.$$

Thus, the total market size also increases.

Surplus obtained by a customer from the low type firm is

$$S_1^* = R - \frac{2\sqrt{\theta}\sqrt{AB}}{r_1} - \frac{(r_2 - r_1)\left(Rr_1 - 4\sqrt{A}\sqrt{B}\sqrt{\alpha} + r_1\,(r_1 - r_2)\,\beta\right)}{r_1\,(4r_2 - r_1)}.$$

$$\text{Thus, } \frac{dS_1^*}{dr_1} = \frac{3Rr_1^2 r_2 + 9\beta r_1^2 r_2^2 + 2\sqrt{AB}\begin{pmatrix} 8\left(\sqrt{\theta} - \sqrt{\alpha}\right) r_2^2 + 8\sqrt{\theta}\,(r_2 - r_1)\,r_2 \\ +\sqrt{\theta} r_1^2 + 2\sqrt{\alpha} r_1\,(2r_2 - r_1) \end{pmatrix}}{r_1^2\,(4r_2 - r_1)^2}. \tag{27}$$

From Equation (27), we find that $\frac{dS_1^*}{dr_1} > 0$ because $\sqrt{\theta} \geq \sqrt{\alpha}$. Thus, surpluses of the retained customers of the low type firm increase. The customers of the high type firm have increased surpluses because price of the high type firm decreases.

$$\frac{d\Pi_1^*}{dr_1} = \frac{Nr_2\left(Rr_1 - 4\sqrt{AB}\sqrt{\alpha}\right)\left(Rr_1 r_2\,(-7r_1 + 4r_2) + 4\,(2r_1^2 - 3r_1 r_2 + 4r_2^2)\,\sqrt{AB}\sqrt{\alpha}\right)}{2r_1^2\,(4r_2 - r_1)^{3}\sqrt{AB}\,(1 - \sqrt{\alpha})}.$$

The profit increases when $\left(Rr_1r_2\left(-7r_1+4r_2\right)+4\left(2r_1^2-3r_1r_2+4r_2^2\right)\sqrt{AB}\sqrt{\alpha}\right)>0$.

$$\frac{d\Pi_2^*}{dr_1}=-\frac{2N\left(r_1+2r_2\right)\left(Rr_2-\sqrt{AB}\sqrt{\alpha}\right)^2}{\left(4r_2-r_1\right)^3\sqrt{AB}\left(1-\sqrt{\alpha}\right)}<0.$$

When firms cross-sell:

By using condition (7), we determine whether the changes in prices, market sizes, and customer surpluses are monotonic or not.

(a)

$$\frac{dp_1^*}{dr_1}=\frac{r_1^2\left(\beta\left(r_1-7r_2\right)\left(r_1-r_2\right)-3Rr_2\right)+4\sqrt{\alpha}\sqrt{AB}\left(r_1^2-2r_1r_2+4r_2^2\right)}{r_1^2\left(4r_2-r_1\right)^2}-\frac{r_2\left(2\sigma_1+\sigma_2\right)}{\left(r_1-4r_2\right)^2} \tag{28}$$

$$\frac{dp_2^*}{dr_1}=\frac{6\sqrt{\alpha}\sqrt{AB}-6Rr_2+\beta\left(r_1^2-8r_1r_2-2r_2^2\right)}{\left(4r_2-r_1\right)^2}-\frac{2r_2\left(2\sigma_1+\sigma_2\right)}{\left(r_1-4r_2\right)^2} \tag{29}$$

From $p_1^*>0$, we have $Rr_1-4\sqrt{AB}\sqrt{\alpha}>r_1(r_2-r_1)\beta+\frac{r_1r_2}{r_2-r_1}(2\sigma_1+\sigma_2)>0$. $\frac{dp_2^*}{dr_1}$ can be rewritten as:

$$\frac{-6(Rr_2-\sqrt{\alpha}\sqrt{AB})}{(4r_2-r_1)^2}-\frac{\beta(2r_2^2+8r_1r_2-r_1^2)+2r_2(2\sigma_1+\sigma_2)}{(4r_2-r_1)^2}.$$

Since $Rr_1>4\sqrt{AB}\sqrt{\alpha}$ and $r_2>r_1$, we have $\frac{dp_2^*}{dr_1}<0$.

The condition for increase in p_1^* can be obtained using $\frac{dp_1^*}{dr_1}>0$. The second order derivative of p_1^* is:

$$\frac{d^2p_1^*}{dr_1^2}=\frac{2\begin{pmatrix}3Rr_1^3r_2+12r_1^2r_2\sqrt{AB}\sqrt{\alpha}-48r_1r_2^2\sqrt{AB}\sqrt{\alpha}+\\64r_2^3\sqrt{AB}\sqrt{\alpha}+r_1^3\left(-4\sqrt{AB}\sqrt{\alpha}+r_2(9r_2\beta+2\sigma_1+\sigma_2)\right)\end{pmatrix}}{r_1^3(r_1-4r_2)^3}<0.$$

This means that the function is concave and has only one stationary point, which we assume as $\tilde{r}_L$.

(b)

$$\begin{aligned}\frac{dM_2^*}{dr_1}=&\left(\frac{Nr_2}{1-\sqrt{\alpha}}\right)\left(\frac{-\sqrt{\alpha}\sqrt{AB}+r_2\left(R+3\beta r_2\right)}{\sqrt{AB}\left(4r_2-r_1\right)^2}\right)\\&-\frac{Nr_2^2\left(-4r_1r_2\sigma_2+r_1^2\left(\sigma_1+\sigma_2\right)+2r_2^2\left(-2\sigma_1+3\sigma_2\right)\right)}{2\sqrt{A}\sqrt{B}\left(-1+\sqrt{\alpha}\right)\left(r_1-4r_2\right)^2\left(r_1-r_2\right)^2}\end{aligned} \tag{30}$$

For $\frac{dM_2^*}{dr_1} > 0$, from the above expression, we find that the following condition should hold

$$r_1^2(2Rr_2 - 2\sqrt{AB\alpha} + r_2(6r_2\beta + \sigma_1 + \sigma_2)) - r_1(4Rr_2^2 - 4r_2\sqrt{AB\alpha} + 4r_2^2(3r_2\beta + \sigma_2)) +$$
$$(2Rr_2^3 - 2\sqrt{AB\alpha}r_2^2 + 2r_2^3(3r_2\beta - 2\sigma_1 + 3\sigma_2)) > 0.$$

Note that the coefficient of r_1 is positive and that is necessary for this proof to work since it plays an important part in the rearrangement we are doing next. Rearranging the terms, we get:

$$\frac{r_1^2(2Rr_2 - 2\sqrt{AB\alpha} + r_2(6r_2\beta + \sigma_1 + \sigma_2)) + (2Rr_2^3 - 2\sqrt{AB\alpha}r_2^2 + 2r_2^3(3r_2\beta - 2\sigma_1 + 3\sigma_2))}{4Rr_2^2 - 4r_2\sqrt{AB\alpha} + 4r_2^2(3r_2\beta + \sigma_2)} - r_1 > 0$$

Suppose $\frac{r_1^2(2Rr_2 - 2\sqrt{AB\alpha} + r_2(6r_2\beta + \sigma_1 + \sigma_2)) + (2Rr_2^3 - 2\sqrt{AB\alpha}r_2^2 + 2r_2^3(3r_2\beta - 2\sigma_1 + 3\sigma_2))}{4Rr_2^2 - 4r_2\sqrt{AB\alpha} + 4r_2^2(3r_2\beta + \sigma_2)} = \Gamma_{2L}^{CS}$. Therefore, M_2^* increases with an increase in r_1 if $\Gamma_{2L}^{CS} - r_1 > 0$ holds.

When (i) $\frac{d\Gamma_{2L}^{CS}}{dr_1} < 1$ and (ii) domain and range of Γ_{2L}^{CS} are same ($X \to X$), Γ_{2L}^{CS} is a contraction mapping. Also, by (ii) and using Banach's fixed point theorem (Kreyszig 2001, pp. 300-304, Fuente 2000, p. 86), we can state that there is one and only one (fixed) point $\hat{r}_1^{M_2}$ such that $\Gamma_{2L}^{CS}(\hat{r}_{M_2}) = \hat{r}_1^{M_2}$. All roots of equation $r_1 = \Gamma_{2L}^{CS}$ are stationary points including $\hat{r}_1^{M_2}$. At $r_1 \to 0$, $\Gamma_{2L}^{CS} > 0$, which means $\frac{dM_2^*}{dr_1} = \Gamma_{2L}^{CS}(r_1) - r_1 > 0$ at $r_1 \to 0$, that is M_2^* is an increasing function in r_1 at $r_1 \to 0$. There are two possibilities since the second order derivative of M_2^* in r_1 is not 0, either M_2^* is convexly or it is concavely increasing in r_1. In the former case, a stationary point cannot be achieved. Only in the latter case a stationary point can be achieved – M_2^* continues to increase at a decreasing rate and reaches at $r_1 = \hat{r}_1^{M_2}$. After that point, M_2^* decreases with a further increase in r_1. Therefore, when $r_1 < \hat{r}_1^{M_1}$, M_2^* increases; otherwise M_2^* decreases.

(c)

$$\frac{dM_1^*}{dr_1} = \frac{Nr_2}{1-\sqrt{\alpha}}\left(\frac{-4\sqrt{\alpha}\sqrt{AB} + 4Rr_2 - \beta\left(r_1^2 - 8r_1r_2 + 4r_2^2\right)}{2\sqrt{AB}\left(4r_2 - r_1\right)^2}\right)$$
$$- \frac{Nr_2^2\left(-8r_1r_2\sigma_1 + 4r_2^2\left(2\sigma_1 - \sigma_2\right) + r_1^2\left(3\sigma_1 + \sigma_2\right)\right)}{2\sqrt{A}\sqrt{B}\left(-1+\sqrt{\alpha}\right)\left(r_1 - 4r_2\right)^2\left(r_1 - r_2\right)^2} \tag{31}$$

Similar to part (b), we rearrange $\frac{dM_1^*}{dr_1} = 0$ into the form $\frac{dM_1^*}{dr_1} = \Gamma_{1L}^{CS}(r_1) - r_1 > 0$ where

$$\Gamma_{1L}^{CS}(r_1) = -\frac{\left(4Rr_2 - 4\sqrt{\mathrm{AB}\alpha}\right)\left(r_1^2 + r_2^2\right) + \beta\left(-r_1^4 + 10r_1^3r_2 - 21r_1^2r_2^2 - 4r_2^4\right) + \sigma_1r_2\left(3r_1^2 + 8r_2^2\right) + \sigma_2r_2\left(r_1^2 - 4r_2^2\right)}{8r_2\left(Rr_2 - \sqrt{AB\alpha} - 2r_2^2\beta + r_2\sigma_1\right)}$$

When (i) $\frac{d\Gamma_{1L}^{CS}(r_1)}{dr_1} < 1$ and (ii) domain and range of $\Gamma_{1L}^{CS}(r_1)$ are same, the rest of the argument is similar to the one shown in the case of M_2^*, that we determine a fixed point $\hat{r}_1^{M_1}$, which is a root of $\frac{dM_1^*}{dr_1} = 0$, such that when $r_1 < \hat{r}_1^{M_1}$ the market size of the high type firm is increasing, otherwise decreasing.

Now, we analyze the full market (M^*).

$$\frac{dM^*}{dr_1} = \frac{Nr_2\left(-6\sqrt{A}\sqrt{B}\sqrt{\alpha} - r_1^2\beta + 2r_2\left(3R + 4r_1\beta + r_2\beta + 2\sigma_1 + \sigma_2\right)\right)}{2\left(r_1 - 4r_2\right)^2\sqrt{A}\sqrt{B}\left(1 - \sqrt{\alpha}\right)} \tag{32}$$

Using the fact that $M_1^* > 0$ and $M_2^* > 0$, we can show $\frac{dM^*}{dr_1} > 0$.

Surplus obtained by a customer from the low type firm is

$$S_1^* = R - \frac{2\sqrt{\theta}\sqrt{AB}}{r_1} - \frac{\left(r_2 - r_1\right)\left(Rr_1 - 4\sqrt{A}\sqrt{B}\sqrt{\alpha} + r_1\left(r_1 - r_2\right)\beta\right)}{r_1\left(4r_2 - r_1\right)} - \frac{r_2\left(2\sigma_1 + \sigma_2\right)}{4r_2 - r_1} + \beta r_1.$$

$$\text{Thus, } \frac{dS_1^*}{dr_1} = \frac{3Rr_1^2r_2 + 9\beta r_1^2r_2^2 + 2\sqrt{AB}\begin{pmatrix} 8\left(\sqrt{\theta} - \sqrt{\alpha}\right)r_2^2 + 8\sqrt{\theta}\left(r_2 - r_1\right)r_2 \\ +\sqrt{\theta}r_1^2 + 2\sqrt{\alpha}r_1\left(2r_2 - r_1\right) \end{pmatrix}}{r_1^2\left(4r_2 - r_1\right)^2} + \frac{r_2\left(2\sigma_1 + \sigma_2\right)}{\left(r_1 - 4r_2\right)^2}. \tag{33}$$

From Equation (33), we find that $\frac{dS^*}{dr_1} > 0$ because $\sqrt{\theta} \geq \sqrt{\alpha}$. Thus, surpluses of the retained customers of the low type firm increase. Surpluses of the customers of the high type firm increases because the high type firm's price decreases. Therefore, everyone's surpluses increase.

Next, we provide the condition when the profit of the low type firm, i.e., Π_1^*, increases when r_1 increases. Suppose

$$\phi_1 = \begin{pmatrix} Rr_1\left(r_1 - r_2\right) + \sigma_2r_1r_2 + \sqrt{AB}\sqrt{\alpha}\left(-4r_1 + 4r_2\right) + \\ \sigma_1\left(r_1^2 - 2r_1r_2\right) + \beta\left(r_1^3 - 2r_1^2r_2 + r_1r_2^2\right) \end{pmatrix}, \text{ and}$$

$$\phi_2(r_1) = \begin{pmatrix} -Rr_1\left(7r_1 - 4r_2\right)\left(r_1 - r_2\right)r_2 + \left(8r_1^3 - 20r_1^2r_2 + 28r_1r_2^2 - 16r_2^3\right)\sqrt{AB}\sqrt{\alpha} \\ +\left(2r_1^5 - 19r_1^4r_2 + 36r_1^3r_2^2 - 23r_1^2r_2^3 + 4r_1r_2^4\right)\beta + \\ \left(-5r_1^3r_2 + 10r_1^2r_2^2 - 8r_1r_2^3\right)\sigma_1 + \left(-2r_1^3r_2 + r_1^2r_2^2 + 4r_1r_2^3\right)\sigma_2 \end{pmatrix}$$

Then,

$$\frac{d\Pi_1^*}{dr_1} = N\frac{\phi_1\phi_2(r_1)}{2(4r_2 - r_1)^3(r_2 - r_1)^2\sqrt{AB}\left(1 - \sqrt{\alpha}\right)}.$$

Now, $\phi_1 = -M_1^* < 0$. $\frac{d\Pi_1^*}{dr_1} > 0$ when $\phi_2 < 0$.

Now we prove that the profit of the high type firm, i.e., Π_2^* decreases when r_1 increases. Suppose

$$\phi_1 = \begin{pmatrix} -2R\left(r_1 - r_2\right)r_2 + \left(2r_1 - 2r_2\right)\sqrt{AB}\sqrt{\alpha} + \\ \left(-r_1^2 r_2 - r_1 r_2^2 + 2r_2^3\right)\beta - r_1 r_2 \sigma_1 + \left(-r_1 r_2 + 2r_2^2\right)\sigma_2 \end{pmatrix}, \text{ and}$$

$$\phi_2 = \begin{pmatrix} -2R\left(r_1 - r_2\right)r_2\left(r_1 + 2r_2\right) + \left(2r_1^2 + 2r_1 r_2 - 4r_2^2\right)\sqrt{AB}\sqrt{\alpha} \\ + \left(r_1^3 r_2 - 15r_1^2 r_2^2 + 18r_1 r_2^3 - 4r_2^4\right)\beta + \left(-r_1^2 r_2 - 4r_1 r_2^2 + 8r_2^3\right)\sigma_1 + \\ \left(-r_1^2 r_2 + 2r_1 r_2^2 - 4r_2^3\right)\sigma_2 \end{pmatrix}$$

Then,

$$\frac{d\Pi_2^*}{dr_1} = -N\frac{\phi_1\phi_2}{2(4r_2 - r_1)^3(r_2 - r_1)^2\sqrt{AB}\left(1 - \sqrt{\alpha}\right)}.$$

We show that $\phi_1 > 0$ and $\phi_2 > 0$. We know that $M_2^* > 0$, and note that $\phi_1 = M_2^*$. Therefore, $\phi_1 > 0$. Next, we use the fact that $p_2^* > 0$ to show that $\phi_2 > 0$. We compare the coefficients of $\sqrt{AB}\sqrt{\alpha}$, β, σ_1, and σ_2 in ϕ_2 and p_2^*. First, we rewrite ϕ_2 as

$$\phi_2 = R - Rev_{C1}\sqrt{AB}\sqrt{\alpha} - Rev_{C2}\beta + Rev_{C3}\sigma_1 - Rev_{C4}\sigma_2, \text{ such that}$$

$$\begin{aligned} Rev_{C1} &= \frac{-\left(2r_1^2 + 2r_1 r_2 - 4r_2^2\right)}{-2\left(r_1 - r_2\right)r_2\left(r_1 + 2r_2\right)},\ Rev_{C2} = \frac{-\left(r_1^3 r_2 - 15r_1^2 r_2^2 + 18r_1 r_2^3 - 4r_2^4\right)}{-2\left(r_1 - r_2\right)r_2\left(r_1 + 2r_2\right)}, \\ Rev_{C3} &= \frac{\left(-r_1^2 r_2 - 4r_1 r_2^2 + 8r_2^3\right)}{2\left(r_2 - r_1\right)r_2\left(r_1 + 2r_2\right)},\ \text{and } Rev_{C4} = \frac{-\left(-r_1^2 r_2 + 2r_1 r_2^2 - 4r_2^3\right)}{-2\left(r_1 - r_2\right)r_2\left(r_1 + 2r_2\right)}. \end{aligned}$$

Likewise, since $p_2^* > 0$, we have

$$R - \underbrace{\frac{4\left(-r_1 + r_2\right)}{r_1\left(r_2 - r_1\right)}}_{P_{C1}}\sqrt{AB}\sqrt{\alpha} - \underbrace{\frac{-\left(-r_1^3 + 2r_1^2 r_2 - r_1 r_2^2\right)}{r_1\left(r_2 - r_1\right)}}_{P_{C2}}\beta - \underbrace{\frac{2r_1 r_2}{r_1\left(r_2 - r_1\right)}}_{P_{C3}}\sigma_1 - \underbrace{\frac{r_1 r_2}{r_1\left(r_2 - r_1\right)}}_{P_{C4}}\sigma_2 > 0.$$

Now,

$$\begin{aligned} P_{C1} - Rev_{C1} &= \frac{4r_2 - r_1}{r_1 r_2} > 0,\ P_{C2} - Rev_{C2} = \frac{3r_1\left(4r_2 - r_1\right)}{2\left(r_1 + 2r_2\right)} > 0,\ \text{and} \\ P_{C4} - Rev_{C4} &= \frac{r_1\left(4r_2 - r_1\right)}{2\left(r_2 - r_1\right)\left(2r_2 + r_1\right)} > 0. \end{aligned}$$

Thus, since $R - P_{C1}\sqrt{AB}\sqrt{\alpha} - P_{C2}\beta - P_{C4}\sigma_2 > P_{C3}\sigma_1 > 0$, then

$\phi_2 = R - Rev_{C1}\sqrt{AB}\sqrt{\alpha} - Rev_{C2}\beta + Rev_{C3}\sigma_1 - Rev_{C4}\sigma_2 > 0$.

The overall aggregate surplus increases because the surpluses of all individual customers increase including the switchers. Since overall market size increases and the surpluses of individual customers increase, the total surplus also increases.

Proof of Proposition 5: The first order derivatives as shown below on equilibrium prices, market sizes of the firms and the customer surplus with respect to r_2 to determine the changes in equilibrium when the high type firm improves its recommender system.

When firms do not cross-sell:

(a) $\frac{dp_1^*}{dr_2} = \frac{3\left(-4\sqrt{A}\sqrt{B}\sqrt{\alpha}+Rr_1\right)}{(r_1-4r_2)^2} > 0$, $\frac{dp_2^*}{dr_2} = \frac{2\sqrt{AB}\sqrt{\alpha}(4r_2(r_2-r_1)+r_1^2)+r_1r_2(6Rr_2-8\sqrt{AB}\sqrt{\alpha})}{r_2^2(4r_2-r_1)^2} > 0$ because $Rr_1 > 4\sqrt{AB}\sqrt{\alpha}$.

(b,c) Using the fact that $M^* > 0$, we can show that $\frac{dM^*}{dr_2} > 0$. Now, $\frac{dM_1^*}{dr_2} = \frac{-Nr_1\left(Rr_1-4\sqrt{A}\sqrt{B}\sqrt{\alpha}\right)}{2\sqrt{A}\sqrt{B}\left(1-\sqrt{\alpha}\right)(r_1-4r_2)^2} < 0$, $\frac{dM_2^*}{dr_2} = \frac{N\left(2R(2r_2-r_1)r_2+\sqrt{A}\sqrt{B}\sqrt{\alpha}r_1\right)}{\sqrt{A}\sqrt{B}\left(1-\sqrt{\alpha}\right)(r_1-4r_2)^2} > 0$.

(d) Surplus obtained by a customer from the high type firm is given by the following expression.

$$CS_2^* = R - \frac{\sqrt{AB}\sqrt{\theta}}{r_2} + \frac{(r_2-r_1)\left(2Rr_2 - 2\sqrt{A}\sqrt{B}\sqrt{\alpha}\right)}{(4r_2-r_1)\,r_2}.$$

$$\frac{dCS_2^*}{dr_2} = -\frac{1}{(r_1-4r_2)^2 r_2^2}\Bigg(2\left(3Rr_1r_2^2 + r_1^2\left(\sqrt{A}\sqrt{B}\sqrt{\alpha} - \sqrt{A}\sqrt{B}\sqrt{\theta}\right)\right.$$
$$\left.\left. -4r_2^2\left(-\sqrt{A}\sqrt{B}\sqrt{\alpha} + 4\sqrt{A}\sqrt{B}\sqrt{\theta}\right) + r_1r_2\left(8\sqrt{A}\sqrt{B}\left(-\sqrt{\alpha}+\sqrt{\theta}\right)\right)\right)\right) \tag{34}$$

Using $\frac{dCS_2^*}{dr_2} > 0$ we can find $\theta > \theta_{ls}^{NCS}$ with higher surpluses.

Profit:

$$\frac{d\Pi_1^*}{dr_2} = \frac{N\left(-4\sqrt{AB}\sqrt{\alpha} + Rr_1\right)^2 (r_1+2r_2)}{2\sqrt{A}\sqrt{B}\,(1-\sqrt{\alpha})\,(4r_2-r_1)^3} > 0.$$

The profit of the high type firm obviously increases because both price and market size increase $\left(\frac{d\Pi_2^*}{dr_2} = p_2^*\frac{dM_2^*}{dr_2} + M_2^*\frac{dp_2^*}{dr_2} > 0\right)$.

When firms cross-sell:

We use the fact that prices and market sizes are positive, and use the condition in Equation (7) to determine whether changes in prices, market sizes, and customer surpluses are monotonic or not.

(a)

$$\frac{dp_1^*}{dr_2} = \frac{-12\sqrt{\alpha}\sqrt{AB} + 3Rr_1 + 2\beta\left(r_1 - r_2\right)\left(r_1 + 2r_2\right)}{\left(4r_2 - r_1\right)^2} + \frac{r_1\left(2\sigma_1 + \sigma_2\right)}{\left(r_1 - 4r_2\right)^2},$$

$$\frac{dp_2^*}{dr_2} = \frac{1}{(4r_2 - r_1)^2}\Bigg(2\sqrt{AB}\sqrt{\alpha}(2r_2 - r_1)^2 + (6Rr_2 - 8\sqrt{AB}\sqrt{\alpha})r_1r_2 +$$

$$\beta r_2^2(8r_2^2 - 4r_1r_2 + 5r_2^2) + 2r_1r_2(2\sigma_1 + \sigma_2)\Bigg).$$

$p_1^* > 0 \implies Rr_1 > 4\sqrt{AB}\sqrt{\alpha}$, and since $r_2 > r_1$, $\frac{dp_2^*}{dr_2} > 0$. Using $\frac{dp_1^*}{dr_2} > 0$ we find the condition for the increase the p_1^*.

(b,c)

$$\frac{dM_2^*}{dr_2} = \left(\frac{N}{1-\sqrt{\alpha}}\right)\left(\frac{\sqrt{\alpha}\sqrt{AB}r_1 + r_2\left(2R\left(2r_2 - r_1\right) + \beta\left(8r_2^2 - r_1r_2 - r_1^2\right)\right)}{\sqrt{AB}\left(4r_2 - r_1\right)^2}\right) -$$

$$\frac{Nr_2\left(20r_1r_2^2\sigma_2 - 8r_2^3\sigma_2 + 2r_1^3\left(\sigma_1 + \sigma_2\right) - r_1^2r_2\left(5\sigma_1 + 11\sigma_2\right)\right)}{2\sqrt{A}\sqrt{B}\left(1-\sqrt{\alpha}\right)\left(r_1 - 4r_2\right)^2\left(r_1 - r_2\right)^2}. \tag{35}$$

Rearrange the equation $\frac{dM_2^*}{dr_2} = 0$ into the form $\frac{dM_2^*}{dr_2} = r_2 - \Gamma_{2H}^{CS}$ where

$$\Gamma_{2H}^{CS} = -2\beta\left(\frac{r_1}{r_2}\right)^4 + \left(-\frac{4R}{r_2} + \frac{2\sqrt{AB}\sqrt{\alpha}}{r_2^2} + 2\beta - \frac{2\left(\sigma_1 + \sigma_2\right)}{r_2}\right)\left(\frac{r_1}{r_2}\right)^3 + \left(\frac{16R}{r_2} - \frac{4\sqrt{AB}\sqrt{\alpha}}{r_2^2}\right.$$

$$\left. + 18\beta + \frac{5\sigma_1 + 11\sigma_2}{r_2}\right)\left(\frac{r_1}{r_2}\right)^2 + \left(-\frac{20R}{r_2} + \frac{2\sqrt{AB}\sqrt{\alpha}}{r_2^2} - 34\beta - \frac{20\sigma_2}{r_2}\right)\left(\frac{r_1}{r_2}\right)$$

$$8\left(\frac{R}{r_2} + \left(2\beta + \frac{\sigma_2}{r_2}\right)\right).$$

When (i) $\frac{\Gamma_{2H}^{CS}}{dr_2} < 1$, and (ii) domain and range of $\Gamma_{2H}^{CS}(r_2)$ are same (i.e., $X \to X$), then Γ_{2H}^{CS} is a contraction mapping. By using (ii) and Banach's fixed point theorem, we can state that there is one and only one (fixed) point $\hat{r}_2^{M_2}$ such that $\Gamma_{2H}^{CS}(\hat{r}_{M_2}) = \hat{r}_2^{M_2}$. All roots of equation $r_2 = \Gamma_{2H}^{CS}$ are stationary points including $\hat{r}_2^{M_2}$.

We provide this proof for $\sigma_2 > \sigma_1$. Then, $M_2^* \to +\infty$ at $r_2 \to r_1$, which means for $r_1 < r_2 < \hat{r}_2^{M_2}$, M_2^* must decrease in order to reach to the stationary point $\hat{r}_2^{M_2}$, after which it should be increasing.

$$\frac{dM_1^*}{dr_2} = -\left(\frac{N}{1-\sqrt{\alpha}}\right)\frac{r_1\left(-4\sqrt{\alpha}\sqrt{AB} + Rr_1 + \beta\left(r_1^2 - 2r_1r_2 + 4r_2^2\right)\right)}{2\sqrt{AB}\left(4r_2 - r_1\right)^2} -$$

$$\frac{Nr_1^2\left(r_1^2\sigma_1 + r_2^2\left(6\sigma_1 - 5\sigma_2\right) + 2r_1r_2\left(-2\sigma_1 + \sigma_2\right)\right)}{2\sqrt{A}\sqrt{B}\left(1-\sqrt{\alpha}\right)\left(r_1 - 4r_2\right)^2\left(r_1 - r_2\right)^2}. \tag{36}$$

We rewrite $\frac{dM_1^*}{dr_2} = \Gamma_{1H}^{CS} - r_2$. When (i) $\frac{\Gamma_{2L}^{CS}}{dr_2} < 1$, and (ii) domain and range of $\Gamma_{2L}^{CS}(r_2)$ are same, as in the previous cases, we can show using Banch's fixed point theorem that $\hat{r}_2^{M_1} = \Gamma_{1H}^{CS}(\hat{r}_2^{M_1})$, where

$$\Gamma_{1H}^{CS} = -\beta\left(\frac{r_1}{r_2}\right)^4 + \left(-\frac{R}{r_2} + 4\beta - \frac{\sigma_1}{r_2}\right)\left(\frac{r_1}{r_2}\right)^3 + \left(\frac{2R}{r_2} + \frac{4\sqrt{AB}\sqrt{\alpha}}{r_2^2} - 9\beta + \frac{4\sigma_1 - 2\sigma_2}{r_2}\right)\left(\frac{r_1}{r_2}\right)^2 + \left(-\frac{R}{r_2} - \frac{8\sqrt{\alpha}}{r_2^2} - 10\beta + \frac{6\sigma_1 - 5\sigma_2}{r_2}\right)\left(\frac{r_1}{r_2}\right) - 4\left(\frac{\left(-5 + 4\sqrt{AB}\right)\sqrt{\alpha}}{r_2^2} + \beta\right).$$

Being a root of $r_2 = \Gamma_{1H}^{CS}$, it is a stationary point. $M_1^* \to -\infty$ when $r_2 \to r_1$, which means the contour of M_1^* must increase when r_1 starts from 0 and increases further in order to reach at the stationary point. Therefore, M_1^* increases when $r_2 < \hat{r}_2^{M_1}$ and decreases thereafter.

$$\frac{dM^*}{dr_2} = \frac{N\begin{pmatrix} R(8r_2^2 - 4r_1r_2 - r_1^2) + \beta(-r_1^3 + 16r_2^3 - 6r_2^2r_1) + \\ 6r_1\sqrt{AB}\sqrt{\alpha} - r_1^2\sigma_1 + 8r_2^2\sigma_2 - 4r_2r_2\sigma_2 \end{pmatrix}}{2\,(r_1 - 4r_2)^2\sqrt{A}\sqrt{B}\,(1 - \sqrt{\alpha})} \tag{37}$$

The full market $M^* = M_1^* + M_2^*$ increases when $\frac{dM^*}{dr_2} > 0$. Here is the proof. We have to show that

$$R(8r_2^2 - 4r_1r_2 - r_1^2) + \beta(-r_1^3 + 16r_2^3 - 6r_2^2r_1) + 6r_1\sqrt{AB}\sqrt{\alpha} - r_1^2\sigma_1 + 8r_2^2\sigma_2 - 4r_2r_2\sigma_2 > 0$$

$$\text{i.e., } R(5r_2 + r_1)(r_2 - r_1) + \beta(14r_2^3 - 5r_1^2r_2 + 2r_1r_2^2) + 6r_1\sqrt{AB}\sqrt{\alpha} + \sigma_2r_2(6r_2 - 5r_1) \underbrace{+3Rr_2^2 + \beta(2r_2^3 - r_1^3 - 2r_1r_2^3 - r_1^2r_2) + \sigma_2r_2(2r_2 + r_1) - r_1^2\sigma_1}_{\text{X}} > 0$$

Now, since $p_1^* > 0$ and $p_2^* > 0$, $p_1^* + p_2^* > 0$, which is expressed as:

$$2Rr_2^2 + \beta(2r_2^3 - r_1^3 - 2r_1r_2^3 - r_1^2r_2) + \sigma_2r_2(2r_2 + r_1) - r_1r_2\sigma_1 > R(r_1^2 + r_1r_2) + 6\sqrt{AB}\sqrt{\alpha}(r_2 - r_1).$$

The above expression implies that

$X = 3Rr_2^2 + \beta(2r_2^3 - r_1^3 - 2r_1r_2^3 - r_1^2r_2) + \sigma_2r_2(2r_2 + r_1) - r_1^2\sigma_1 > 0$. Thus, $\frac{dM^*}{dr_2} > 0$.

From Equations (36) and (35) we get the conditions for increase in the market sizes using $\frac{dM_1^*}{dr_2} > 0$ and $\frac{dM_2^*}{dr_2} > 0$.

Surplus obtained by a customer from the high type firm is given by the following expression.

$$CS_2^* = R - \frac{\sqrt{AB}\sqrt{\theta}}{r_2} + \frac{(r_2 - r_1)\left(2Rr_2 - 2\sqrt{A}\sqrt{B}\sqrt{\alpha} + r_2\,(r_1 + 2r_2)\,\beta\right)}{(4r_2 - r_1)\,r_2} - \frac{r_1\sigma_1 + 2r_2\sigma_2}{4r_2 - r_1} + \beta r_2.$$

$$\frac{dCS_2^*}{dr_2} = -\frac{1}{(r_1-4r_2)^2 r_2^2}\Big(2\Big(3Rr_1r_2^2 + r_1^2\left(\sqrt{A}\sqrt{B}\sqrt{\alpha} + 2r_2^2\beta - \sqrt{A}\sqrt{B}\sqrt{\theta}\right)$$
$$-4r_2^2\left(-\sqrt{A}\sqrt{B}\sqrt{\alpha} + r_2^2\beta + 4\sqrt{A}\sqrt{B}\sqrt{\theta}\right) + r_1r_2\left(8\sqrt{A}\sqrt{B}\left(-\sqrt{\alpha}+\sqrt{\theta}\right) + r_2\left(2r_2\beta + 2\sigma_1 + \sigma_2\right)\right)\Big)\Big) \tag{38}$$

The condition $\theta > \theta_{ls}^{CS}$ for customers that have increased surpluses can be obtained by solving $\frac{dCS_2^*}{dr_2} > 0$ for $\sqrt{\theta}$.

Next, we provide the condition when the profit of the low type firm, i.e., Π_1^*, increases when r_2 increases. Suppose

$$\phi_1 = \begin{pmatrix} Rr_1(r_1-r_2) + \sigma_2 r_1 r_2 + \sqrt{AB}\sqrt{\alpha}\,(-4r_1+4r_2) + \\ \sigma_1(r_1^2 - 2r_1r_2) + \beta(r_1^3 - 2r_1^2r_2 + r_1r_2^2) \end{pmatrix}, \text{ and}$$

$$\phi_2 = \begin{pmatrix} 2R(r_1-r_2)(2r_1^2 - 3r_1r_2 + 4r_2^2) + \sqrt{AB}\sqrt{\alpha}\,(-14r_1^2 + 22r_1r_2 - 8r_2^2) \\ +\beta(2r_1^4 + 3r_1^3r_2 - 27r_1^2r_2^2 + 46r_1r_2^3 - 24r_2^4) + (2r_1^3 - r_1^2r_2 - 4r_1r_2^2)\sigma_1 + \\ (2r_1^3 - 9r_1^2r_2 + 18r_1r_2^2 - 8r_2^3)\sigma_2 \end{pmatrix}$$

Then,

$$\frac{d\Pi_1^*}{dr_2} = \frac{N\phi_1\phi_2}{2(4r_2-r_1)^3(r_2-r_1)^2\sqrt{AB}\,(1-\sqrt{\alpha})}.$$

Now, $\phi_1 = -M_1^* < 0$. $\frac{d\Pi_1^*}{dr_2} > 0$ when $\phi_2 < 0$.

Now, we prove that the profit of the high type firm, i.e., Π_2^*, decreases when r_2 increases. Suppose

$$\phi_1 = \begin{pmatrix} -2R(r_1-r_2)r_2 + (2r_1 - 2r_2)\sqrt{AB}\sqrt{\alpha} + \\ (-r_1^2r_2 - r_1r_2^2 + 2r_2^3)\beta - r_1r_2\sigma_1 + (-r_1r_2 + 2r_2^2)\sigma_2 \end{pmatrix}, \text{ and}$$

$$\phi_2 = \begin{pmatrix} -2R(r_1-r_2)(2r_1^2 - 3r_1r_2 + 4r_2^2) + (14r_1^2 - 22r_1r_2 + 8r_2^2)\sqrt{AB}\sqrt{\alpha} \\ +(-2r_1^4 - 3r_1^3r_2 + 27r_1^2r_2^2 - 46r_1r_2^3 + 24r_2^4)\beta + \\ (-2r_1^3 + r_1^2r_2 + 4r_1r_2^2)\sigma_1 + (-2r_1^3 + 9r_1^2r_2 - 18r_1r_2^2 + 8r_2^3)\sigma_2 \end{pmatrix}$$

Then,

$$\frac{d\Pi_2^*}{dr_2} = \frac{N\phi_1\phi_2}{2(4r_2-r_1)^3(r_2-r_1)^2\sqrt{AB}\,(1-\sqrt{\alpha})}.$$

We show that $\phi_1 > 0$ and $\phi_2 > 0$. We know that $M_2^* = \phi_1 > 0$. Next, we use the fact that $p_2^* > 0$ to show that $\phi_2 > 0$. We compare the coefficients of $\sqrt{AB}\sqrt{\alpha}$, β, σ_1, and σ_2 in ϕ_2 and p_2^*. First, we rewrite ϕ_2 as

$$\phi_2 = R - (Rev_{C1}\sqrt{AB}\sqrt{\alpha} + Rev_{C2}\beta + Rev_{C4}\sigma_2) + Rev_{C3}\sigma_1, \text{ such that}$$

$$\begin{aligned} Rev_{C1} &= \frac{-\left(14r_1^2 - 22r_1r_2 + 8r_2^2\right)}{2\left(r_2 - r_1\right)\left(2r_1^2 - 3r_1r_2 + 4r_2^2\right)},\ Rev_{C2} = \frac{-\left(-2r_1^4 - 3r_1^3r_2 + 27r_1^2r_2^2 - 46r_1r_2^3 + 24r_2^4\right)}{2\left(r_2 - r_1\right)\left(2r_1^2 - 3r_1r_2 + 4r_2^2\right)}, \\ Rev_{C3} &= \frac{\left(-2r_1^3 + r_1^2r_2 + 4r_1r_2^2\right)}{2\left(r_2 - r_1\right)\left(2r_1^2 - 3r_1r_2 + 4r_2^2\right)}, \text{ and } Rev_{C4} = \frac{-\left(-2r_1^3 + 9r_1^2r_2 - 18r_1r_2^2 + 8r_2^3\right)}{2\left(r_2 - r_1\right)\left(2r_1^2 - 3r_1r_2 + 4r_2^2\right)}. \end{aligned}$$

Likewise, since $p_2^* > 0$, we have

$$R - \left(\underbrace{\frac{4}{r_1}}_{P_{C1}}\sqrt{AB}\sqrt{\alpha} + \underbrace{(r_2 - r_1)}_{P_{C2}}\beta + \underbrace{\frac{r_2}{(r_2 - r_1)}}_{P_{C4}}\sigma_2\right) > \frac{2r_1r_2}{r_1\left(r_2 - r_1\right)}\sigma_1 > 0.$$

Now,

$$\begin{aligned} P_{C1} - Rev_{C1} &= \frac{\left(4r_2 - r_1\right)^2}{r_1\left(2r_1^2 - 3r_1r_2 + 4r_2^2\right)} > 0,\ P_{C2} - Rev_{C2} = \frac{\left(4r_2 - r_1\right)\left(2r_1^2 - 7r_1r_2 + 8r_2^2\right)}{4r_1^2 - 6r_1r_2 + 8r_2^2} > 0, \text{ and} \\ P_{C4} - Rev_{C4} &= \frac{\left(4r_2 - r_1\right)\left(\left(r_2 - r_1\right)^2 + 3r_2\left(r_2 - r_1\right) + r_1^2\right)}{2\left(r_2 - r_1\right)\left(2r_1^2 - 3r_1r_2 + 4r_2^2\right)} > 0. \end{aligned}$$

Thus, $\phi_2 > 0$.

The equations for aggregate surpluses from the low type and the high type firms, respectively, are the following:

$$\begin{aligned} \Sigma_1^* = {} & \frac{1}{2r_1\left(r_1 - 4r_2\right)^2\left(r_1 - r_2\right)^2\sqrt{AB}}\Big(r_2\Big(5Rr_1\left(r_1 - r_2\right)r_2 + 4r_2^2\sqrt{AB}\sqrt{\alpha} + 3r_1^3r_2\beta \\ & -r_1r_2^2\left(3r_2\beta + 2\sigma_1 + 3\sigma_2\right) + r_1^2\left(-4\sqrt{AB}\sqrt{\alpha} + 3r_2\sigma_1 + 2r_2\sigma_2\right)\Big)\Big(Rr_1\left(r_1 - r_2\right) + 4r_2\sqrt{AB}\sqrt{\alpha} \\ & +r_1^3\beta + r_1^2\left(-2r_2\beta + \sigma_1\right) + r_1\left(-4\sqrt{AB}\sqrt{\alpha} + r_2\left(r_2\beta - 2\sigma_1 + \sigma_2\right)\right)\Big)\Big), \text{ and} \end{aligned} \tag{39}$$

$$\begin{aligned} \Sigma_2^* = {} & \frac{1}{2\left(r_1 - 4r_2\right)^2\left(r_1 - r_2\right)^2\sqrt{AB}}\Big(r_2\Big(2R\left(r_1 - r_2\right)r_2 + r_1^2r_2\beta - 2r_2 \\ & \left(-\sqrt{AB}\sqrt{\alpha} + r_2\left(r_2\beta + \sigma_2\right)\right) + r_1\left(-2\sqrt{AB}\sqrt{\alpha} + r_2\left(r_2\beta + \sigma_1 + \sigma_2\right)\right)\Big)^2\Big). \end{aligned} \tag{40}$$

Finally,

$$\Phi_H = \frac{d\Sigma_1^*}{dr_2} + \frac{d\Sigma_2^*}{dr_2}. \tag{41}$$

Proof of Proposition 6: Using $\frac{dp^*}{dr} = \frac{\beta}{2} + \frac{\sqrt{AB}(\sqrt{\alpha} - \delta(1 - \sqrt{\alpha}))}{r^2} < 0$, we find the condition for the decrease in price with the increase in r.

E. Equilibrium for Recommender System Effectivenesses in a Monopoly and a Duopoly

E.1. Monopoly

In this section, we determine the recommender system equilibrium when there are variable costs associated with providing particular recommender system effectivenesses. We introduce now a cost of providing a specific recommendation effectiveness, and assume that the costs are convex in nature. The costs incurred by the firm include the cost of developing the infrastructure for storing, analyzing, and maintaining the customer data (Leavitt 2006). The cost of developing infrastructure is incurred only once. Some other one time costs may be incurred by the firm while improving the system. We assume that the firm has the capital for building the infrastructure, bears the one time costs and the costs are sunk. We consider the cost of maintaining the recommender system and providing recommendations of a certain level of effectiveness as cr^2.

E.1.1. When firms cross-sell The profit equation of the firm is

$$\Pi = \frac{N(p+\sigma)}{(1-\sqrt{\alpha})}\left(\sqrt{\bar{\theta}}-\sqrt{\alpha}\right) - cr^2. \tag{42}$$

The above equation is solved to determine optimal price first and the optimal recommender system effectiveness next. The price p^* is same as the one determined in the main analysis. The optimal price is used to recalculate the profit equation and using the first order condition in r, the r^* is determined, which is the solution of the following equation.

$$\begin{aligned} &-3N\beta^2 r^4 + \left(16c\sqrt{AB}\left(-1+\sqrt{\alpha}\right) - 4NR\beta - 4N\beta\sigma\right) r^3 + \\ &\left(-NR^2 + 4N\sqrt{AB}\sqrt{\alpha}\beta - 2NR\sigma - N\sigma^2\right) r^2 + 4NAB\alpha = 0. \end{aligned} \tag{43}$$

The value of r^* is profit maximizing if

$$c > \frac{N\left(4\mathrm{AB}\alpha + r^3\beta(2R+3r\beta+2\sigma)\right)}{8r^3\sqrt{AB}\left(1-\sqrt{\alpha}\right)},$$

which is obtained from the second order condition.

E.1.2. When firms do not cross-sell In this case, the optimal r^* is a solution of the following equation

$$16\sqrt{AB}c\left(-1+\sqrt{\alpha}\right)r^3+NR^2r^2+4ABN\alpha=0,$$

which is a profit maximizing effectiveness value when

$$c>\frac{\sqrt{AB}N\alpha}{2r^3\left(1-\sqrt{\alpha}\right)}.$$

E.2. Duopoly

As in the previous subsection, the costs for providing the effectivenesses r_1 and r_2 are $c_1r_1^2$ and $c_2r_2^2$, respectively. To determine the price and recommender system effectiveness equilibrium, we consider a two stage game. In the first stage, the recommender system effectiveness is decided through a simultaneous move game because that is likely to be a relatively permanent decision. Then the firms play a simultaneous move price game. Both firms may end up changing their system effectivenesses when the cost parameters (c_1 and c_2).

E.2.1. When firms cross-sell The profit equations of the two firms are as follows.

$$\Pi_1=D_1(p_1,p_2,r_1,r_2)(p_1+\sigma_1)-c_1r_1^2 \text{ and } \Pi_2=D_2(p_1,p_2,r_1,r_2)(p_2+\sigma_2-c_2r_2^2, \text{where}$$

$$D_1(p_1,p_2,r_1,r_2)=\int_{\alpha}^{\theta_s}\frac{N}{2(1-\sqrt{\alpha})\sqrt{\theta}}d\theta=N\frac{(\sqrt{\theta_s}-\sqrt{\alpha})}{1-\sqrt{\alpha}} \text{ and}$$

$$D_2(p_1,p_2,r_1,r_2)p_2=\int_{\theta_s}^{\bar{\theta}_2}\frac{N}{2(1-\sqrt{\alpha})\sqrt{\theta}}d\theta=N\frac{\left(\sqrt{\bar{\theta}_2}-\sqrt{\theta_s}\right)}{1-\sqrt{\alpha}}.$$

We find the equilibrium prices as we find in the main model. After substituting the equilibrium prices in the above equations, we simultaneously solve $\frac{\partial\Pi_1}{\partial r_1}=0$ and $\frac{\partial\Pi_2}{\partial r_2}=0$ to determine equilibrium r_1 and r_2. Following equations represent this system of equations with two variables.

$$-4\sqrt{A}\sqrt{B}\left(-1+\sqrt{\alpha}\right)c_1r_1^{*3}\left(r_1^*-4r_2^*\right)^3\left(r_1^*-r_2^*\right)^2+Nr_2^*\Big(\beta r_1^{*3}+4\sqrt{A}\sqrt{B}\sqrt{\alpha}r_2^*+r_1^{*2}\left(R-2\beta r_2^*+\sigma_1\right)+$$

$$r_1^*\left(-4\sqrt{A}\sqrt{B}\sqrt{\alpha}+r_2^*\left(-R+\beta r_2^*-2\sigma_1+\sigma_2\right)\right)\Big)\Big(2\beta r_1^{*5}-19\beta r_1^{*4}r_2^*-16\sqrt{A}\sqrt{B}\sqrt{\alpha}r_2^{*3}+$$

$$r_1^{*3}\Big(8\sqrt{A}\sqrt{B}\sqrt{\alpha}+$$

$$r_2^* \left(-7R + 36\beta r_2^* - 5\sigma_1 - 2\sigma_2\right)) + 4r_1^* r_2^{*2} \left(7\sqrt{A}\sqrt{B}\sqrt{\alpha} + r_2^* \left(-R + \beta r_2^* - 2\sigma_1 + \sigma_2\right)\right) +$$
$$r_1^{*2} r_2^* \left(-20\sqrt{A}\sqrt{B}\sqrt{\alpha} + r_2^* \left(11R - 23\beta r_2^* + 10\sigma_1 + \sigma_2\right)\right)\Big) = 0 \tag{44}$$

$$2N\beta^2 r_1^{*6} r_2^* + 16 \Big(-ABN\alpha r_2^{*3} + 3N\beta^2 r_2^{*7} + N r_2^{*5} \left(R^2 - 2\sqrt{A}\sqrt{B}\sqrt{\alpha}\beta + 2R\sigma_2 + \sigma_2^2\right) +$$
$$4r_2^{*6} \left(4\sqrt{A}\sqrt{B}\left(-1 + \sqrt{\alpha}\right)c_2 + N\beta\left(R + \sigma_2\right)\right)\Big) + 2r_1^{*4} \Big(-14N\beta^2 r_2^{*3} - 2\sqrt{A}\sqrt{B}N\sqrt{\alpha}\left(2R + \sigma_1 + \sigma_2\right) +$$
$$N r_2^* \left(4\left(R^2 - 2\sqrt{A}\sqrt{B}\sqrt{\alpha}\beta\right) + \left(\sigma_1 + \sigma_2\right)\left(4R + \sigma_1 + \sigma_2\right)\right) - 2r_2^{*2} \Big(-14\sqrt{A}\sqrt{B}\left(-1 + \sqrt{\alpha}\right)c_2 +$$
$$N\beta\left(R - \sigma_1 + 2\sigma_2\right))) + r_1^{*3} \Big(28ABN\alpha + 13N\beta^2 r_2^{*4} + 8\sqrt{A}\sqrt{B}N\sqrt{\alpha} r_2^* \left(-\sigma_1 + \sigma_2\right) - N r_2^{*2}$$
$$\left(28R^2 - 68\sqrt{A}\sqrt{B}\sqrt{\alpha}\beta + \sigma_1^2 + 40R\sigma_2 + 13\sigma_2^2 + 2\sigma_1\left(8R + 7\sigma_2\right)\right) - 4r_2^{*3} \Big(73\sqrt{A}\sqrt{B}\left(-1 + \sqrt{\alpha}\right)c_2 +$$
$$N\beta\left(16R + 9\sigma_1 + 7\sigma_2\right))) - 4r_1^* r_2^{*2} \Big(-15ABN\alpha + 29N\beta^2 r_2^{*4} + 4\sqrt{A}\sqrt{B}N\sqrt{\alpha} r_2^* \left(R + \sigma_1\right) +$$
$$11N r_2^{*2} \left(R^2 - 2\sqrt{A}\sqrt{B}\sqrt{\alpha}\beta + 2R\sigma_2 + \sigma_2^2\right) + 4r_2^{*3} \left(44\sqrt{A}\sqrt{B}\left(-1 + \sqrt{\alpha}\right)c_2 + N\beta\left(11R + \sigma_1 + 10\sigma_2\right)\right)\Big) +$$
$$4r_1^{*2} r_2^* \Big(-18ABN\alpha + 19N\beta^2 r_2^{*4} + \sqrt{A}\sqrt{B}N\sqrt{\alpha} r_2^* \left(6R + 7\sigma_1 - \sigma_2\right) + N r_2^{*2} \Big(12R^2 - 26\sqrt{A}\sqrt{B}\sqrt{\alpha}\beta - \sigma_1^2 +$$
$$22R\sigma_2 + 9\sigma_2^2 + 2\sigma_1\left(R + 2\sigma_2\right)) + r_2^{*3} \left(172\sqrt{A}\sqrt{B}\left(-1 + \sqrt{\alpha}\right)c_2 + N\beta\left(43R + 11\sigma_1 + 32\sigma_2\right)\right)\Big) +$$
$$r_1^{*5} \left(-4\sqrt{A}\sqrt{B}N\sqrt{\alpha}\beta + r_2^* \left(-4\sqrt{A}\sqrt{B}\left(-1 + \sqrt{\alpha}\right)c_2 + N\beta\left(5\beta r_2^* + 4\left(2R + \sigma_1 + \sigma_2\right)\right)\right)\right) = 0. \tag{45}$$

Although the equations to determine the equilibrium r_1^* and r_2^* are intractable, we numerically find the solutions of these equations for a wide range of values of the parameters. Based on those solutions, in Figure 4 we show how the prices and recommender system effectivenesses change when c_1 changes, and in Figure (5) we show how prices and recommender system effectivenesses change when c_2 changes.

E.2.2. When firms do not cross-sell Using the same steps described in the previous subsection for the two stage game, we find the following simultaneous equations that should be solved for determining the equilibrium r_1 and r_2 when firms do not cross-sell.

$$4\sqrt{AB}\left(-1 + \sqrt{\alpha}\right)c_1 r_1^6 + 48\sqrt{AB}\left(-1 + \sqrt{\alpha}\right)c_1 r_1^5 r_2 + 48\left(-ABN\alpha + 2\sqrt{A}\sqrt{B}\sqrt{AB}N\alpha\right)r_1 r_2^2$$
$$- 192\sqrt{AB}\left(-1 + \sqrt{\alpha}\right)c_1 r_1^4 r_2^2 - 64\left(-ABN\alpha + 2\sqrt{A}\sqrt{B}\sqrt{AB}N\alpha\right)r_2^3 +$$
$$r_1^3 \Big(-8ABN\alpha + 8\sqrt{A}\sqrt{B}\sqrt{AB}N\alpha + 2\left(7\sqrt{A}\sqrt{B} - 3\sqrt{AB}\right)NR\sqrt{\alpha} r_2 - 7NR^2 r_2^2 +$$

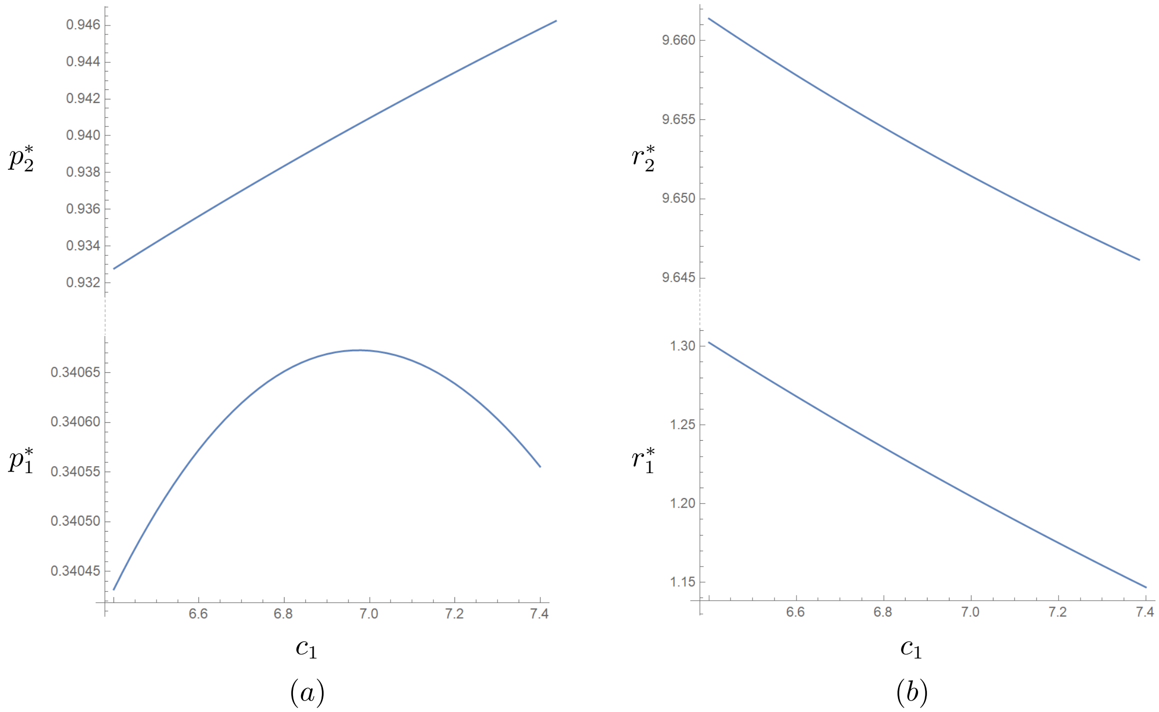

Figure 4 Variations in r_1^*, r_2^*, p_1^*, and p_2^* with change in c_1 ($n=10000$, $A=50$, $B=50$, $R=2$, $\alpha=0.000001$, $c_2=8$, $\beta=0.02$, $\sigma_1=0.018$, and $\sigma_2=0.100$)

$$256\sqrt{AB}\left(-1+\sqrt{\alpha}\right)c_1r_2^3\Big)+4Nr_1^2r_2\Big(4AB\alpha-12\sqrt{A}\sqrt{B}\sqrt{AB}\alpha$$

$$+Rr_2\left(2\left(-\sqrt{A}\sqrt{B}+3\sqrt{AB}\right)\sqrt{\alpha}+Rr_2\right)\Big)=0 \tag{46}$$

$$7ABN\alpha r_1-2\sqrt{A}\sqrt{B}NR\sqrt{\alpha}r_1^2+\Big(-4ABN\alpha-4\sqrt{A}\sqrt{B}NR\sqrt{\alpha}r_1+2NR^2r_1^2-$$

$$\sqrt{AB}\left(-1+\sqrt{\alpha}\right)c_2r_1^3\Big)r_2+\left(-3NR^2r_1+12\sqrt{AB}\left(-1+\sqrt{\alpha}\right)c_2r_1^2\right)r_2^2+$$

$$\left(4NR^2-48\sqrt{AB}\left(-1+\sqrt{\alpha}\right)c_2r_1\right)r_2^3+64\sqrt{AB}\left(-1+\sqrt{\alpha}\right)c_2r_2^4=0 \tag{47}$$

We numerically find that when c_1 decreases, both firms improve their recommender system effectivenesses, and the high type firm decreases its price (p_2^*). When c_2 decreases, both firms again improve their recommender system effectivenesses and the high type firm increases its price.

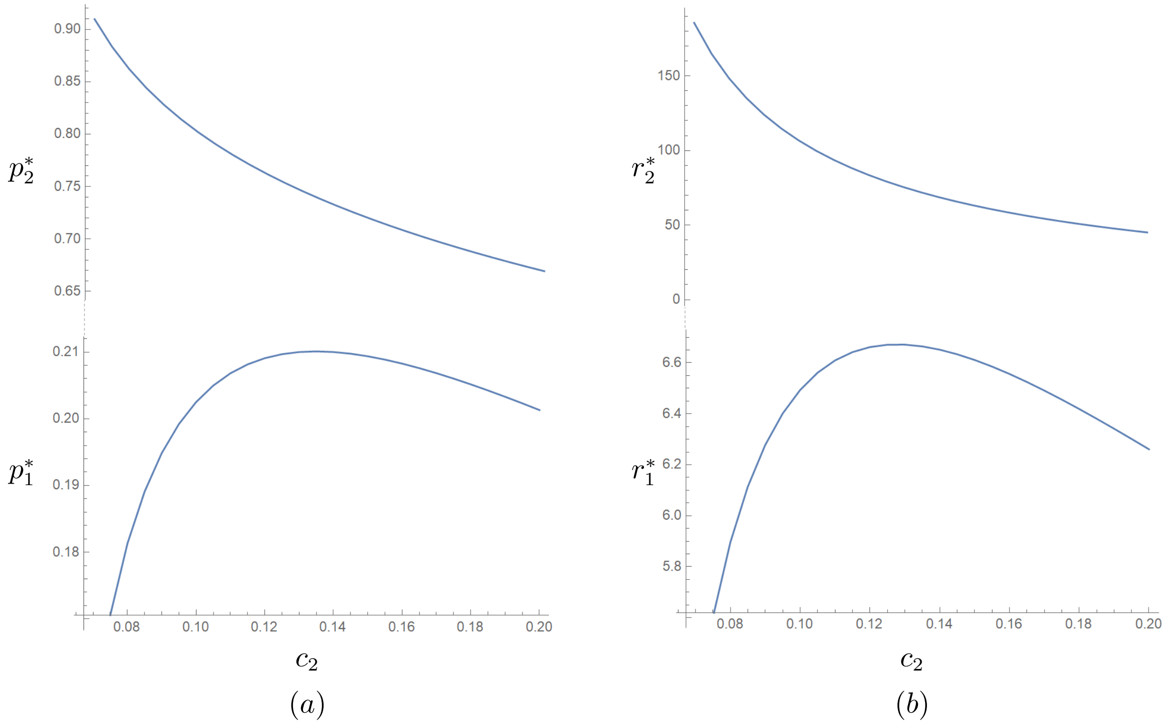

Figure 5 Variations in r_1^*, r_2^*, p_1^*, and p_2^* with change in c_2 ($n = 10000$, $A = 500$, $B = 500$, $R = 2$, $\alpha = 0.000001$, $c_1 = 0.08$, $\beta = 0.002$, $\sigma_1 = 0.051$, and $\sigma_2 = 0.300$)

F. Analysis of the case when Price of Cross-selling product and Focal Product are Same

In practice, it is possible that a firm obtains its cross-selling revenue by selling its own products that are similar to the focal product. Consequently, the price of such cross-sold products are likely to be same as (or correlated with) the price of the focal product. We now analyze the markets with monopoly and duopoly for such a scenario.

F.1. All products cross-sold are same as focal products

F.1.1. Monopoly As we did previously, we solve the decision problem of the customer and then that of the firm. We consider that the price of the cross-sold product sold by the firm is same as that of the focal product (if the price is not same but correlated, the analysis and the results will not change). Thus, the new cross-selling surplus is $(\omega_0(R - \frac{A(1+\omega)}{yr^2} - p))$. The parameter ω_0 is the probability of purchasing the cross-selling product of the firm, and $(R - \frac{A(1+\omega)}{yr^2} - p)$ denotes the utility derived from the cross-selling product of the firm (note that the customer should not now incur the search cost, but the reservation price and the mismatch cost should be the same as that of the focal product). Hence the new surplus equation for a customer is

$$\begin{aligned} S &= (1+\omega)R - \frac{A(1+\omega)}{yr^2} - B\theta y - p - \omega p \\ &= \tilde{R} - \frac{\tilde{A}}{yr^2} - B\theta y - p - \omega p, \end{aligned}$$

where $\tilde{R} = (1+\omega)R$ and $\tilde{A} = (1+\omega)A$. The customer decides the optimal effort to determine her maximum surplus (S^*) and searches and purchases a product if S^* is non-negative.

The revenue of the firm is,

$$\Pi = \underbrace{\frac{N(p)}{2(1-\sqrt{\alpha})} \int_{\alpha}^{\bar{\theta}} \frac{1}{\sqrt{\theta}} d\theta}_{\text{Revenue from focal product}} + \underbrace{\frac{N(\omega p)}{2(1-\sqrt{\alpha})} \int_{\alpha}^{\bar{\theta}} \frac{1}{\sqrt{\theta}} d\theta}_{\substack{\text{Revenue from cross-selling} \\ \text{own products}}} = \frac{N(1+\omega)p}{(1-\sqrt{\alpha})} \left(\sqrt{\bar{\theta}} - \sqrt{\alpha}\right).$$

In the above equations of surplus and profit, on substituting $\tilde{p} = p(1+\omega)$, we get

$$S = \tilde{R} - \frac{\tilde{A}}{yr^2} - B\theta y - \tilde{p} \text{ and } \Pi = \frac{N\tilde{p}}{(1-\sqrt{\alpha})} \left(\sqrt{\bar{\theta}} - \sqrt{\alpha}\right).$$

Now, the decision variable of the firm is $\tilde{p}$ instead of p. With this substitution, the equations structurally and characteristically remain identical to the ones we have already analyzed for the case when firms do not cross-sell.

F.1.2. Duopoly As we did in the analysis of monopoly, we revise the surplus equation of the customer as

$$S_i^* = \tilde{R} - \frac{\tilde{A}}{y_i r_i^2} - B\theta y_i - p_i - \omega p_i,$$

where ωp_i is the expected price of the product of firm $i \in \{1, 2\}$ for cross-selling, $\tilde{R} = (1+\omega)R$, and $\tilde{A} = (1+\omega)A$. The profit of the low type and the high types firms, respectively, are

$$\begin{aligned} \Pi_1 &= (p_1 + \omega p_1) \int_{\alpha}^{\theta_s} \frac{N}{2(1-\sqrt{\alpha})\sqrt{\theta}} d\theta = N(1+\omega)p_1 \frac{(\sqrt{\theta_s} - \sqrt{\alpha})}{1-\sqrt{\alpha}} \text{ and} \\ \Pi_2 &= (p_2 + \omega p_2) \int_{\theta_s}^{\bar{\theta}_2} \frac{N}{2(1-\sqrt{\alpha})\sqrt{\theta}} d\theta = N(1+\omega)p_2 \frac{\left(\sqrt{\bar{\theta}_2} - \sqrt{\theta_s}\right)}{1-\sqrt{\alpha}}. \end{aligned}$$

Again, with the substitution $(1+\omega)p_i = \tilde{p}_i$ we get the surplus and profit equations that are structurally and characteristically same as the ones analyzed in the main model when firms do not cross-sell, with the difference that now $\tilde{p}_i$ are decision variables for the two firms.

F.2. Some cross-sold products are same as focal products and others are sold by third parties

F.2.1. Monopoly As we did previously, we solve the decision problem of the customer and then that of the firm. The cross-selling surplus of the customer is now a combination of surplus obtained from the products sold by the third party and the product sold by the firm. We consider that the price of the cross-sold product sold by the firm is same as that of the focal product (if the price is not same but correlated, the analysis and the results will not change). Thus, the new cross-selling surplus is $(\beta_0 r + \omega(R - \frac{A(1+\omega)}{yr^2} - p))$. The parameter ω is the probability of purchasing the cross-selling product of the firm, and $(R - \frac{A(1+\omega)}{yr^2} - p)$ denotes the utility derived from the cross-selling product of the firm. Hence the new surplus equation for a customer is

$$\begin{aligned} S &= (1+\omega)R - \frac{A(1+\omega)}{yr^2} - B\theta y - p + \beta_0 r - \omega p \\ &= \tilde{R} - \frac{\tilde{A}}{yr^2} - B\theta y - p + \beta_0 r - \omega p, \end{aligned}$$

where $\tilde{R} = (1+\omega)R$ and $\tilde{A} = (1+\omega)A$. The customer decides the optimal effort to determine her maximum surplus (S^*) and searches and purchases a product if $S^* \geq 0$.

Accordingly, the firm obtains the cross-selling revenue partly from selling its own products (for an expected price ωp to each customer) and the rest from the third parties (σ_0). Thus,

$$\Pi = \underbrace{\frac{N(p)}{2\left(1-\sqrt{\alpha}\right)} \int_{\alpha}^{\bar{\theta}} \frac{1}{\sqrt{\theta}} d\theta}_{\text{Revenue from focal product}} + \underbrace{\frac{N(\sigma_0)}{2\left(1-\sqrt{\alpha}\right)} \int_{\alpha}^{\bar{\theta}} \frac{1}{\sqrt{\theta}} d\theta}_{\text{Revenue from parters}} + \underbrace{\frac{N(\omega p)}{2\left(1-\sqrt{\alpha}\right)} \int_{\alpha}^{\bar{\theta}} \frac{1}{\sqrt{\theta}} d\theta}_{\substack{\text{Revenue from cross-selling} \\ \text{own products}}}.$$

The new profit equation of the firm is

$$\Pi = \frac{N(p(1+\omega) + \sigma_0)}{(1-\sqrt{\alpha})} \left(\sqrt{\bar{\theta}} - \sqrt{\alpha}\right).$$

In the above equations of surplus and profit, on substituting $\tilde{p} = p(1+\omega)$, we get

$$S = \tilde{R} - \frac{\tilde{A}}{yr^2} - B\theta y - \tilde{p} + \beta_0 r \text{ and } \Pi = \frac{N(\tilde{p}+\sigma_0)}{(1-\sqrt{\alpha})}\left(\sqrt{\bar{\theta}} - \sqrt{\alpha}\right).$$

Now, the decision variable of the firm is $\tilde{p}$ instead of p. With this substitution, the equations structurally and characteristically remain identical to the ones we have already analyzed. Thus, all the analyses and results remain the same for this relaxation about the cross-selling surplus.

F.2.2. Duopoly As we did in the analysis of monopoly, we revise the surplus equation of the customer as

$$S_i^* = \tilde{R} - \frac{\tilde{A}}{y_i r_i^2} - B\theta y_i - p_i + \beta_0 r_i - \omega p_i,$$

where ωp_i is the expected price of the product of firm $i \in \{1,2\}$. Suppose the expected revenues of the firms that they obtain from the partners is σ_i. The profit of the low type and the high types firms, respectively, are

$$\Pi_1 = (p_1 + \sigma_1 + \omega p_1)\int_{\alpha}^{\theta_s} \frac{N}{2(1-\sqrt{\alpha})\sqrt{\theta}} d\theta = N\left((1+\omega)p_1 + \sigma_1\right)\frac{(\sqrt{\theta_s} - \sqrt{\alpha})}{1-\sqrt{\alpha}} \text{ and}$$

$$\Pi_2 = (p_2 + \sigma_2 + \omega p_2)\int_{\theta_s}^{\bar{\theta}_2} \frac{N}{2(1-\sqrt{\alpha})\sqrt{\theta}} d\theta = N((1+\omega)p_2 + \sigma_2)\frac{\left(\sqrt{\bar{\theta}_2} - \sqrt{\theta_s}\right)}{1-\sqrt{\alpha}}.$$

Again, with the substitution $(1+\omega)p_i = \tilde{p}_i$ we get the surplus and profit equations that are structurally and characteristically same as the ones analyzed in the main model, with the difference that now $\tilde{p}_i$ are decision variables for the two firms. Thus, our analyses and results are not affected.

References

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